Lecture 7

ELTE
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Outline of the lecture

1. Introduction
1.1 Recall: factorization

2. Perturbation theory
2.1 Fixed-order expansion

3. NLO corrections
3.1 Structure of NLO corrections
3.2 Infrared divergences
3.3 Loop calculations
3.4 Current status of NLO automation

4. Electroweak NLO corrections
4.1 Complete-NLO expansion
4.2 Input scheme dependence
4.3 Sudakov enhancement
4.4 Current status of NLO EW automation

5. Shortly on NNLO: the future prospects
5.1 Structure of NNLO corrections

6. Summary

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Recall factorization of hadron collisions

$$\sigma(h_1 h_2 \to X_n) = \sum_{a,b} \int \mathrm{d}x_a \mathrm{d}x_b f_{a/h_1}(x_a, \mu_F) f_{b/h_2}(x_b, \mu_F) \hat{\sigma}_{ab\to X}(\mu_F, \alpha_s(\mu_R)) \mathrm{d}\Phi_n$$

- Recall: parton density functions $f_i(x, \mu_F)$
- Recall: parton-level cross section $\hat{\sigma}$
- Recall: factorization scale μ_F separates long-distance physics and short-distance physics
- **Note:** μ_F**-dependence** appears in the parton-level cross section
- **Renormalisation scale** μ_R appears through the couplig constant $\alpha_S(\mu_R)$ when the theory needs to be renormalized
- Both μ_R and μ_F are **unphysical scales**

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Perturbation theory in QFT

- We cannot solve the quantum field theory of the Standard Model!
- Use **perturbation theory** to find an approximation:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$$

- Recall from particle physics that the SM Lagrangian has the strong (QCD) interaction part:

$$g_s \overline{\Psi} \frac{\lambda^a}{2} G_\mu^a \Psi$$

- The interaction part is proportional to the **coupling strength** $g_s = \sqrt{4\pi\alpha_S}$
- If this coupling constant is $g_s \ll 1$, then we can use **Taylor expansion** of the interaction term:

$$\text{interaction piece} \sim g_s A + g_s^2 B + g_s^3 C + \mathcal{O}(g_s^4)$$

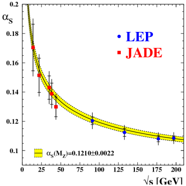
Partonic cross section

Now we can use perturbation theory to **expand the partonic cross section in the strong coupling constant**:

$$\hat{\sigma} = \alpha_s^B \left(\underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s \sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2 \sigma_2}_{\text{NNLO}} + \underbrace{\alpha_s^3 \sigma_3}_{\text{N}^3\text{LO}} + \mathcal{O}(\alpha_s^4) \right)$$

The strong coupling constant is scale-dependent (**renormalization scale**): $\alpha_s(\mu_R)$

- In regions where $\alpha_s \ll 1$ (around $\mu_R \approx 200$ MeV) does not hold $\rightarrow$ **breakdown of perturbation theory!**
- At LHC energies $\sqrt{s} \sim M_Z$, the strong coupling value is $\alpha_S(M_Z) = 0.118$ $\rightarrow \sim 10\%$ correction expected for NLO QCD



If all-order terms are included, the dependence on μ_R vanishes
$\rightarrow$ in practice, the higher order terms included, the better the reliability!

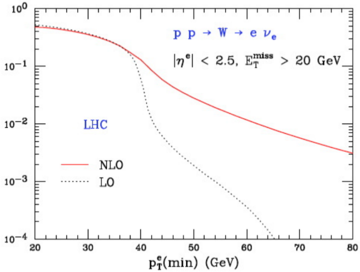
Not everything that looks like NLO is NLO

Be careful! This expansion is valid observable-by-observable, not for an entire process!

Example: $pp \to t\bar{t}$ **at** $\mathcal{O}(\alpha_s^3)$

- $m(t\bar{t})$ is NLO at this order
- $p_T(t)$ is NLO at this order
- $p_T(j)$ is LO at this order (not defined at $\mathcal{O}(\alpha_s^2)$)
- $p_T(t\bar{t})$ is LO at this order

Such large K-factors (=NLO/LO) indicate that the observable is not defined at LO:



Questions?

Question for you:

Consider again:
$pp \to t\bar{t}$ at $\mathcal{O}(\alpha_s^3)$: is the azimuthal distance between the top quarks described at NLO if we run an NLO code?

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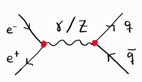
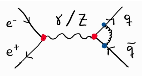
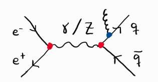
Structure of NLO corrections

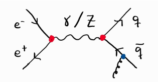
Example: $e^+e^- \to q\overline{q}$

- Leading-order diagram (Born-level): 

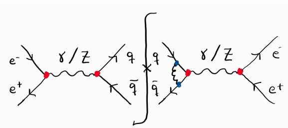
- Possible diagrams with at most g_s^2 order higher:

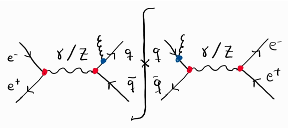






- How can these be combined in $\sigma \sim |\mathcal{M}|^2$ to give at most $\alpha_S \sim g_s^2$ order higher?
[Interference only between same final states!]





Structure of NLO corrections

Challenges

These two types of interferences are called:

- **Virtual corrections** (interference between Born diagram and one-loop diagram)
- **Real-emission correction** (interference between two one-emission diagrams)

The **cross section at NLO** is the sum of these contributions, but the phase-space multiplicity is different!

$$\sigma^{\mathrm{NLO}} = \int_n \mathrm{d}\sigma_B + \int_n \mathrm{d}\sigma_V + \int_{n+1} \mathrm{d}\sigma_R$$

→ in principle we can just integrate these contributions separately!

Seems quite easy: what is then the problem?

THERE ARE TWO PROBLEMS:

1. Infrared divergence
2. Loop calculations

Structure of NLO corrections

Challenges

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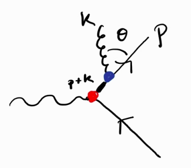
Structure of NLO corrections

Infrared divergences

Consider the real-emission diagram:



The denominator of the **virtual propagator** is

$$(p+k)^2 = p^2 + k^2 + 2p \cdot k = 2(E_p E_k - \overline{p} \cdot \overline{k}) = 2E_p E_k (1 - \cos\theta)$$

Divergence when

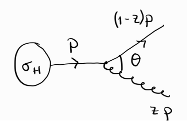
$$E_k \to 0, \quad \theta \to 0$$

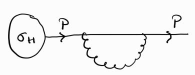
Infrared divergence:
low-energy *or* long-distance divergence

Infrared divergences

Good news: cancellation of IR singularities

Let us consider a similar situation as we saw yesterday in L6: the splitting and a corresponding virtual emission:





- The real emission part is

$$\sigma_R \sim \sigma_H \frac{\alpha_S C_F}{2\pi} \frac{\mathrm{d}z}{z} \frac{\mathrm{d}k_T^2}{k_T^2} \frac{\mathrm{d}\phi}{2\pi}$$

  Note now the **explicit appearance of** C_F **and** z in the denominator (the role of the splitting functions from L6)!

- The virtual emission can be computed to be

$$\sigma_V \sim -\sigma_H \frac{\alpha_S C_F}{2\pi} \frac{\mathrm{d}z}{z} \frac{\mathrm{d}k_T^2}{k_T^2} \frac{\mathrm{d}\phi}{2\pi}$$

- The singularity poles (z and k_T^2) cancel!

Infrared divergences

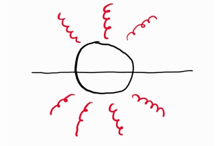
The KLN theorem

The **Kinoshita-Lee-Nauernberg (KLN) theorem** is a statement that the cancellation of singularities happens whenever the observable in question is **infrared safe**

Infrared-safe (IR-safe) observables:
observables which are insensitive to the emission of soft and/or collinear emissions!

$$\mathcal{O}(p_1,\ldots,p_k,\ldots) \to \mathcal{O}(p_1,\ldots \not{p_k},\ldots) \qquad \text{if} \quad E_k \to 0$$
$$\mathcal{O}(p_1,\ldots,p_k,p_n,\ldots) \to \mathcal{O}(p_1,\ldots p_k+p_n,) \qquad \text{if} \quad p_k||p_n$$

Example: number of gluons in event is NOT IR-safe:

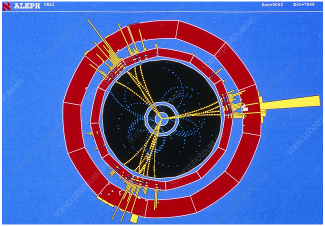


Infrared divergences

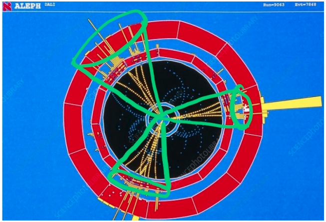
Jet algorithms

To avoid infrared-unsafe observables, one often makes a duality between partons and objects called **jets**:

partons ↔ jets



→



This defines **jet algorithms**:
cluster partons which are **close in phase space**

Infrared divergences

Jet algorithms

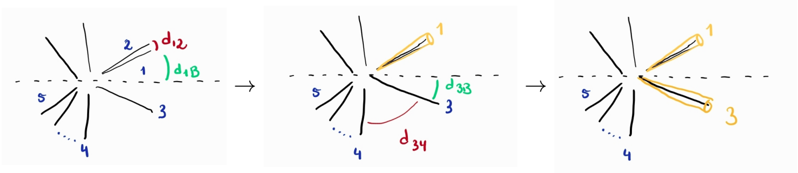
Sequential clustering with radius R and exponent p
For each parton:

1. Compute the distances d_{ij} between every other parton j
2. Compute d_{iB} between parton and beam
3. If $d_{ij} < d_{iB}$, cluster partons i and j
4. If $d_{ij} > d_{iB}$, define i as jet
5. Add new jet to list and remove parton from list

$$d_{ij} = \min(k_{T,i}^p, k_{T,j}^p)\frac{\Delta_{ij}^2}{R^2}$$

$$d_{iB} = k_{T,i}^p$$

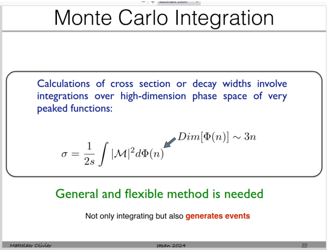
When all partons are removed from list, we are left with clustered jets!



Infrared divergences

The actual cancellation of singularities

- The KLN theorem guarantees cancellation of singularities: but how does this happen **in practice**?
- Integrals are solved with Monte Carlo tools in particle physics:



- The cancellation of the divergences must occur **numerically** for a stable computation!
- Two main approaches exist today:

1. Slicing method
2. Subtraction method

Infrared divergences

Toy example

Consider the divergent one-dimensional integral as toy example:

$$\int_0^1 \mathrm{d}x \frac{f(x)}{x}$$

To make sense of this divergent integral, introduce the parameter ϵ and define the integral as the limit

$$\lim_{\epsilon \to 0} \int_0^1 \mathrm{d}x \frac{f(x)}{x^{1-\epsilon}}$$

since the integral diverges, the pole will appear as $\frac{1}{\epsilon}$

Infrared divergences

Slicing method

In the **slicing approach**, the interval of integration is split with a small parameter δ:

$$\lim_{\epsilon\to 0}\int_0^1 \mathrm{d}x \frac{f(x)}{x^{1-\epsilon}} = \lim_{\epsilon\to 0}\int_0^\delta \mathrm{d}x \frac{f(x)}{x^{1-\epsilon}} + \lim_{\epsilon\to 0}\int_\delta^1 \mathrm{d}x \frac{f(x)}{x^{1-\epsilon}}$$

$$\approx \lim_{\epsilon\to 0} f(0)\int_0^\delta \mathrm{d}x \frac{1}{x^{1-\epsilon}} + \lim_{\epsilon\to 0}\int_\delta^1 \mathrm{d}x \frac{f(x)}{x^{1-\epsilon}}$$

$$= \lim_{\epsilon\to 0} f(0)\frac{\delta^\epsilon}{\epsilon} + \lim_{\epsilon\to 0}\int_\delta^1 \mathrm{d}x \frac{f(x)}{x^{1-\epsilon}}$$

Finally one uses Taylor expansion of $\frac{\delta^\epsilon}{\epsilon}$ around $\epsilon = 0$:

$$\lim_{\epsilon\to 0} f(0)\left(\frac{1}{\epsilon} + \log\delta\right) + \lim_{\epsilon\to 0}\int_\delta^1 \mathrm{d}x \frac{f(x)}{x^{1-\epsilon}}$$

Since the pole appears in the real and virtual in the same way with opposite signs, they can be canceled directly

Infrared divergences

Subtraction method

In the **subtraction approach**, starting from the integral

$$\lim_{\epsilon\to 0}\int \mathrm{d}x\,\frac{f(x)}{x^{1-\epsilon}}$$

add $0 = \frac{f(0)}{x^{1-\epsilon}} - \frac{f(0)}{x^{1-\epsilon}}$:

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_0^1 \mathrm{d}x\,\frac{f(x)}{x^{1-\epsilon}} &= \lim_{\epsilon\to 0}\int_0^1 \mathrm{d}x\left(\frac{f(x)}{x^{1-\epsilon}} + \frac{f(0)}{x^{1-\epsilon}} - \frac{f(0)}{x^{1-\epsilon}}\right)\\
&\approx \lim_{\epsilon\to 0} f(0)\int_0^1 \mathrm{d}x\,\frac{1}{x^{1-\epsilon}} + \lim_{\epsilon\to 0}\int_0^1 \mathrm{d}x\,\frac{f(x)-f(0)}{x^{1-\epsilon}}\\
&= \lim_{\epsilon\to 0} f(0)\left(\frac{1}{\epsilon}\right) + \underbrace{\lim_{\epsilon\to 0}\int_0^1 \mathrm{d}x\,\frac{f(x)-f(0)}{x^{1-\epsilon}}}_{\text{can be integrated}}
\end{aligned}$$

Infrared divergences

Comparison

Slicing method

$$f(0)\left(\frac{1}{\epsilon} + \log\delta\right) + \int_\delta^1 \mathrm{d}x \frac{f(x)}{x^{1-\epsilon}}$$

- ○ First used for $e^+e^- \to 3j$ production [Brandenburg, Uwer, 10.1016/S0550-3213(97)00790-6]
- ✓ Pole is extracted analytically and canceled with virtual part
- × Dependence on δ-parameter
- × Cancellation between large numbers

Subtraction method

$$f(0)\left(\frac{1}{\epsilon}\right) + \int_0^1 \mathrm{d}x \frac{f(x) - f(0)}{x^{1-\epsilon}}$$

- ○ First used for $e^+e^- \to 3j$ production [Ellis, Ross, Terrano, 10.1016/0550-3213(81)90165-6]
- ✓ Pole is extracted analytically and cancelled with virtual part
- ✓ Exact rewriting
- × Cancellation between large numbers
- ✓ Used in MG5_aMC

Infrared divergences

Subtraction for NLO computations

Recall general NLO corrections:

$$\sigma^{\rm NLO} = \int_n \mathrm{d}\sigma_B + \int_n \mathrm{d}\sigma_V + \int_{n+1} \mathrm{d}\sigma_R$$

The additional complexity from the toy example: now we have the two divergences in two different integrals with different phase spaces (R and V)!
Now add the term:

$$\int_{n+1} \mathrm{d}\sigma_C - \int_{n+1} \mathrm{d}\sigma_C$$

one to the real and one to the virtual part:

$$\sigma^{\rm NLO} = \int_n \mathrm{d}\sigma_B + \int_n \left(\mathrm{d}\sigma_V + \int_1 \mathrm{d}\sigma_C\right) + \int_{n+1} (\mathrm{d}\sigma_R - \mathrm{d}\sigma_C)$$

Such that:

- It has the same **pointwise singular behaviour** as R in the limit $\epsilon \to 0$
- It has to be **analytically integrable** in the one-particle phase space: $\int_1 \mathrm{d}\sigma_C$ in d-dimensions (used for the one-loop integration)

Infrared divergences

Subtraction for NLO computations: FKS subtraction

The subtraction in MG5_aMC:
the **FKS subtraction** [Frixione, Kunszt, Signer, hep-ph/9512328]

One more complexity is that there are two types of singularities in the real part

$$\mathrm{d}\sigma_R = |\mathcal{M}_{n+1}|^2 \mathrm{d}\Phi_{n+1}$$

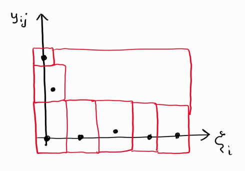
- Soft $\xi_i := E_i\sqrt{s}$
- Collinear: $y_{ij} := \cos\theta_{ij}$

$$|\mathcal{M}_{n+1}|^2 \sim \sum_{ij} \frac{1}{\xi^2} \frac{1}{1-y_{ij}}$$

Split the phase space with an *S*-**function** in a way that each region contains at most one soft and one collinear divergence:

$$\mathrm{d}\sigma_R = \sum_{ij} S_{ij} |\mathcal{M}_{n+1}|^2 \mathrm{d}\Phi_{n+1}$$

$$\sum_{ij} S_{ij} = 1$$



Infrared divergences

Subtraction for NLO computations: FKS subtraction

Recall:

$$\sigma^{\mathrm{NLO}} = \int_n \mathrm{d}\sigma_B + \int_n \left(\mathrm{d}\sigma_V + \int_1 \mathrm{d}\sigma_C \right) + \int_{n+1} (\mathrm{d}\sigma_R - \mathrm{d}\sigma_C)$$

The $\mathrm{d}\sigma_C$ terms contain **counter-events**, which cancel the singularities in the real event

Each real ($(n+1)$-body) event can have a counter-event, which is the corresponding n-body event which cancels:

- Soft: $\xi_i \to 0$
- Collinear: $y_{ij} \to 0$
- Soft and collinear: $\xi_i \to 0$ and $y_{ij} \to 0$

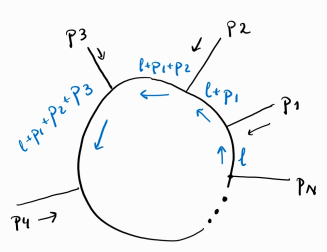
Loop calculations

Now let us consider the computation of the one-loop diagrams:

$$\sigma^{\text{NLO}} = \ldots + \int_n \left(d\sigma_V + \int_1 d\sigma_C \right) + \ldots$$

A general one-loop integral is written as

$$\mathcal{M}^{\text{1-loop}} \sim \int d^4 l \frac{N(l)}{D_1 D_2 \ldots D_m}$$



where the numerator $N(l)$ can be some complicated function of the loop momentum l

Widely spread technique (Veltman-Passarino reduction) for evaluating the loop integral is to cast it in the form

$$\int d^4 l \frac{N(l)}{D_1 D_2 \ldots D_m} = \sum_i C_i \int d^4 l \frac{1}{D_{i,1} D_{i,2} \ldots D_{i,m}}$$

sum of a (unknown) number of scalar integrals

Loop calculations

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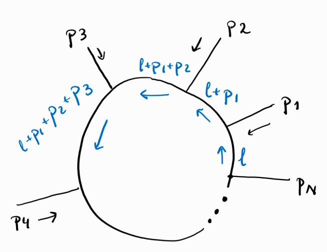

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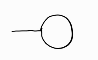
sum of a (unknown) number of scalar integrals

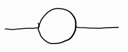
Loop calculations

Good news: if we use $d = 4 + \epsilon$ (which is what we do in our spacetime and in dimensional regularisation!), only terms with up to 4 denominators are needed:

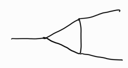
$$\mathcal{M}^{1-\mathrm{loop}} = \sum_{i_0 i_1 i_2 i_3} d_{i_0 i_1 i_2 i_3} \int \mathrm{d}^d l \frac{1}{D_{i_0} D_{i_1} D_{i_2} D_{i_3}}$$



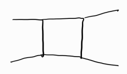
$$+ \sum_{i_0 i_1 i_2} c_{i_0 i_1 i_2 i_3} \int \mathrm{d}^d l \frac{1}{D_{i_0} D_{i_1} D_{i_2}}$$



$$+ \sum_{i_0 i_1} b_{i_0 i_1 i_2 i_3} \int \mathrm{d}^d l \frac{1}{D_{i_0} D_{i_1}}$$



$$+ \sum_{i_0} a_{i_0 i_1 i_2 i_3} \int \mathrm{d}^d l \frac{1}{D_{i_0}}$$



Loop calculations

Evaluating the loop integrals

Scalar loop integrals are universal and are available in libraries:

- FF [van Oldenborgh, CPC 66,1991]
- QCDLoops [Ellis, Zanderighi, arXiv:0712.1851]
- OneLOops [van Hameren, arXiv:1007.4716]

Methods for evaluating the coefficients have been developed:

- Tensor reduction [Denner, Dittmaier, hep-ph/509141]
- Generalized unitarity [Bern et al., hep-ph/9403226]
- **Integrand reduction** (OPP reduction) [Ossola, Papadopoulos, Pittau, hep-ph/0609007]

Current status of NLO automation

NLO computations have become the new standard.

For any non-exotic process (Standard Model processes with *simple final state*), NLO has become the minimal criteria:

Herwig: automated NLO is default
Sherpa: NLO QCD and NLO EW both automated
Pythia: interfaced with multiple matrix-element generators with NLO QCD corrections

Questions?

Question for you:

Is the “hardest jet” observable in an event an IR-safe observable?

Question for home:

How many soft or collinear regions are there to be separated in the FKS subtraction scheme for 3-jet production ($pp \to 3j$)? Count only which are separately soft *or* collinear.

(Note that one can also have soft/collinear singularity with the initial partons! You can have a look at the paper: arXiv:hep-ph/9512328 for further discussion of this process!)

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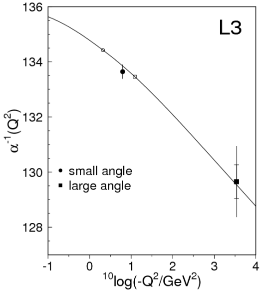
5.1 Structure of NNLO corrections

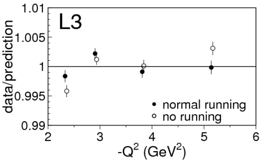
6. Summary

Electroweak force in the SM

$$\underbrace{SU(3)_c}_{\text{strong}} \times \underbrace{SU(2)_W \times U(1)_Y}_{\text{electroweak}}$$

- The Standard Model comprises three fundamental forces:
  *strong force
  *weak force
  *electromagnetic force
- Weak $\oplus$ Electromagnetic $\rightarrow$ **electroweak (EW) force** (coupling g)
- In principle the weak coupling α also depends on μ_R: but this dependence is much milder!
- In practice, one takes a **fixed** α **value** when considering electroweak vertices





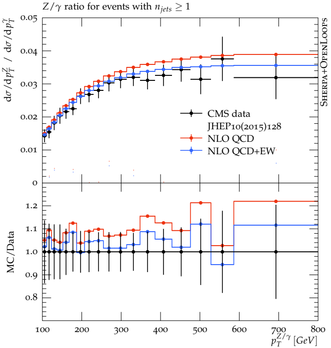
Electroweak corrections: are they needed?

Electroweak corrections are typically small, but they can become important in processes where the experimental error is small!

Example: The data/theory comparison for the ratio

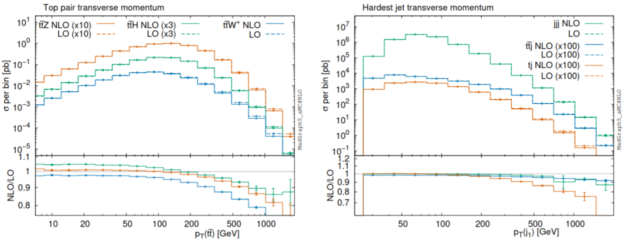
$$\sigma(pp \to Z + jets)/\sigma(pp \to \gamma + jets)$$

between theory (Sherpa) and CMS [Campbell,Ellis,Williams, arXiv:1703.10109]



Size of electroweak corrections

Numerically, $\alpha \sim 0.007$ and $\alpha_S^2 \sim 0.014$ and so typically NLO EW $\sim$ NNLO QCD



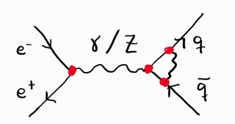
The expected behaviour can be exceeded in certain cases:

- When **photon-induced processes** become relevant and important
- In the **vicinity of decay product invariant masses**, QED radiation might distort differential distributions
- In tails of distributions, **Sudakov enhancement** can become very large

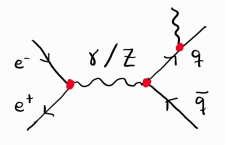
Complete-NLO expansion

NLO electroweak corrections appear as at most **two powers of the electroweak coupling:** g^2

Diagrams with electroweak boson loops:



Diagrams with real emission of electroweak bosons:

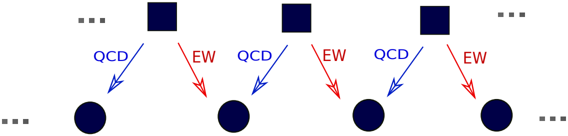


Complete-NLO expansion

If considering both QCD and EW interaction, the perturbative expansion must be done simultaneously in α_S and α:

$$\sigma_{\rm LO} = \alpha_s^n \alpha^m \sigma_{n,m} + \alpha_S^{n-1}\alpha^{m+1}\sigma_{n-1,m+1} + \ldots \alpha_S^{n_0}\alpha^{m_0}\sigma_{n_0 m_0}$$
$$\sigma_{\rm NLO\ QCD+EW} = \alpha_S^{n+1}\alpha^m\sigma_{n+1,m} + \alpha_s^n\alpha^{m+1}\sigma_{n,m+1} + \ldots \alpha_s^{n_0}\alpha^{m_0+1}\sigma_{n_0,m_0+1}$$

Portray this with "blobs" (the number of these is process-dependent):



Note: photons can be considered as jets in the final state!

Complete-NLO expansion

The LO terms in the expansion:

$$\mathrm{LO} = \mathrm{LO}_1 + \mathrm{LO}_2 + \ldots + \mathrm{LO}_k$$

with a process-dependent upper limit k

The NLO terms are:

$$\mathrm{NLO} = \mathrm{NLO}_1 + \mathrm{NLO}_2 + \ldots + \mathrm{NLO}_{k+1}$$

Commonly, NLO QCD is the NLO_1 contribution (the first blob on the NLO-line)
The "NLO EW" corrections are historically the NLO_2 term (second blob on the NLO-line)

Nowadays, with complete-NLO automation, one has to also take into consideration the mixed QCD-EW effects, i.e. the terms NLO_i, i$>$ 1:

$$\mathrm{NLO}_i = \mathrm{LO}_{i-1} \otimes \mathrm{EW} + \mathrm{LO}_i \otimes \mathrm{QCD}$$

Complete-NLO expansion

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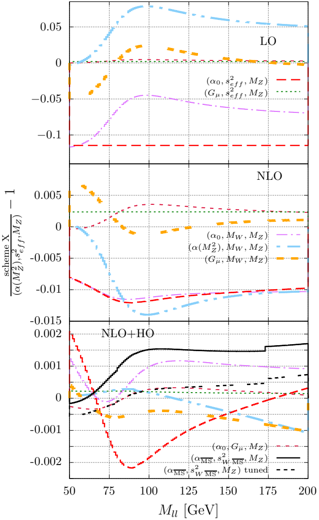
Input schemes



- There are several free parameters in the Standard Model in the electroweak sector, but only **3 are independent!**
- Which three are chosen as input parameters classify the various **input schemes**, most common is to use M_Z and M_W as two of these parameters, and the third one can be:
  - $\alpha(0)$ [the $\alpha(0)$-scheme]
  - $\alpha(M_Z)$ [$\alpha(M_Z)$-scheme]
  - G_μ [G_μ-scheme]

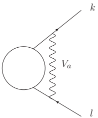
  The relative sizes for the coupling constant in the various input schemes ranges between 2-6% differences [Chiesa et al., arXiv:2402.14659]
- Commonly for LHC, the G_μ-scheme is used
- However, if final-state photons are considered, then the $\alpha(0)$-scheme must be used [Pagani, Tsinikos, Zaro, arXiv:2106.02059]

Sudakov enhancement

The emitted particles in EW can be **massive bosons** (W, Z bosonss)

For massive real emission, the **cancellation between real and virtual contributions is not exact** anymore $\rightarrow$ a significant finite remainder is left!

$$\sum_{k}\sum_{k\neq l}\sum_{V_a=\gamma,Z,W^\pm}$$



In addition, we do not consider the real emission of massive vector bosons together with the virtual emission of vector bosons!
In other words, Sudakov enhancement is the **virtual massive boson exchanges**

These terms are universal and can be computed directly from the Born-level kinematics $\rightarrow$ **At one-loop, we have the Denner-Pozzorini algorithm** [Denner, Pozzorini, hep-ph/0010201]

Sudakov enhancement

Sudakov enhancements appear as **enhanced logarithms** in the high-energy region
$\rightarrow$ electroweak Sudakov logarithms (EWSL)

The order of these corrections at one-loop level are:

$$\mathcal{O}(\mathrm{EWSL}) \sim \left(\alpha \log^k\left(\frac{s}{M_W^2}\right)\right) \times \mathcal{O}(\mathrm{LO})$$

as double-logarithms (DL) for $k = 2$ and single-logarithms (SL) $k = 1$

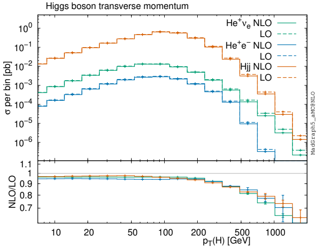
These corrections are universal and appear at matrix-element level as

$$\begin{aligned} \mathcal{M}^{\mathrm{LO+EWSL}} &= \mathcal{M}_0 + \mathcal{M}_0 \times \delta^{\mathrm{EWSL}} \\ \delta^{\mathrm{EWSL}} &= \underbrace{\delta^{\mathrm{LSC}}}_{\mathrm{DL}} + \underbrace{\delta^{\mathrm{SSC}}}_{\mathrm{SL}} + \underbrace{\delta^{\mathrm{C}}}_{\mathrm{SL}} + \underbrace{\delta^{\mathrm{PR}}}_{\mathrm{SL}} \end{aligned}$$

Current status of NLO EW corrections

- The complete-NLO corrections are automated:
  MG5_aMC [Frederix et al., arxiv:1804.10017]
  Sherpa [Kallweit et al., arxiv:1412.5157]

- Two still persisting drawbacks:
  1. no automated matching to parton shower
  2. In general time-consuming computations



Sudakov approximation

Automation

- The Denner-Pozzorini algorithm was first included in Sherpa [Bothmann, Napoletano, arXiv:2006.14635]
- A revision of the algorithm was done and included in MG5_aMC [Pagani, Zaro, arXiv:2110.03714]
- Also an automation in OpenLoops has been performed [Lindert, Mai, arXiv:2312.07927]
- Also, a matching to parton shower, allowing for an approximation of

$$\text{NLO QCD+EW} + \text{parton shower}$$
$$\downarrow$$
$$\text{NLO QCD} \otimes \text{EW Sudakov+parton shower}$$

in MG5_aMC [Pagani, Vitos, Zaro, arXiv:2309.00452] and Sherpa [Bothmann et al., arXiv:2111.13453]

Questions?

Question for you:

In general, in which situations can one find large(r) electroweak corrections?

Question for home:

How many LO "blobs" and how many NLO "blobs" are there for the process $pp \to ZZ + j$?

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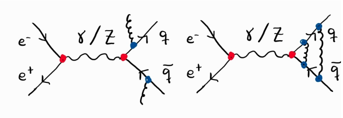
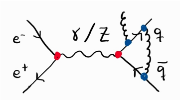
Expanding to NNLO QCD corrections

Example: $e^+e^- \to q\bar{q}$

Diagrams we can obtain with up to g_s^2 orders **higher than real or virtual diagrams** we used for NLO:

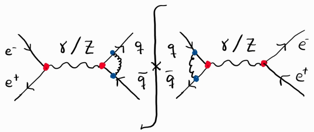




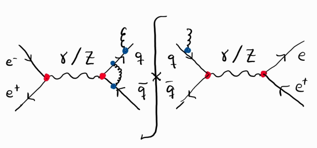
Expanding to NNLO QCD corrections

Interferences we can construct out of these:

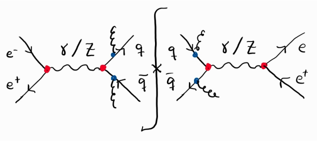
Virtual-**V**irtual:



Real-**V**irtual:



Real-**R**eal:



Expanding to NNLO QCD corrections

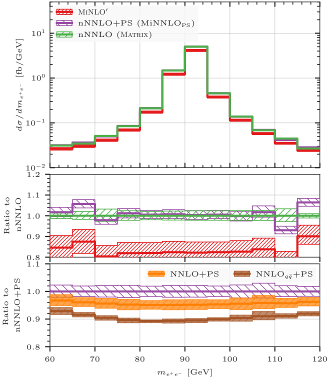
$$\sigma^{\mathrm{NNLO}} = \int_n \mathrm{d}\sigma_{VV} + \int_{n+1} \mathrm{d}\sigma_{RV} + \int_{n+2} \mathrm{d}\sigma_{RR}$$

These are all separately divergent!
Key in NNLO development:

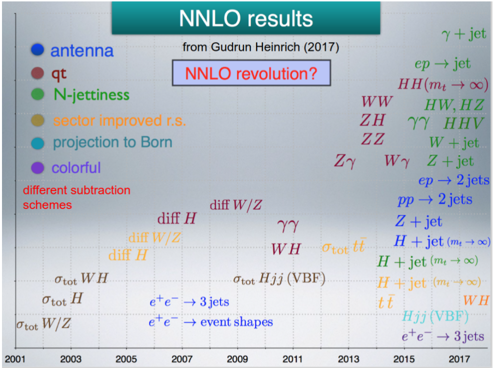
- General formalism for **subtraction scheme**
- Two-loop calculations in general setup

Matching to parton showers:

- No general matching scheme yet
- Process-dependent: MiNNLO$_{\mathrm{PS}}$ [Buonocore et al., arXiv:2108.05337]



Recent progresses in NNLO



- NNLO corrections computed for many processes already!
- Still, a process-independent fully-automated toolbox is still yet to come!

Questions?

Question for you:

What are the three parts to an NNLO computation?

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Summary of today's lecture

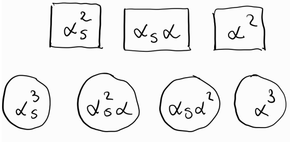
- Perturbation theory in the SM: expansion in QCD coupling as well as a mixed-coupling (QCD+EW) expansion
- Details of NLO computations:
  - Subtraction schemes for IR divergences
  - Loop calculations
- NLO electroweak corrections:
  - Typically not very large and NNLO QCD $\sim$ NLO EW
  - Sudakov enhancement in tails can become large
  - Be attentive to which input scheme one is using
- NNLO corrections:
  - Methodology similar as in NLO corrections, but now with 3 sectors to treat IR divergences: RR, RV, VV
  - **Most prospects in the field of precision phenomenology today**

Final questions for this lecture?

Back-up slides

MG5_aMC syntax for complete-NLO computations

As example, consider $pp \to t\bar{t}$:



To generate the LO_1:

```
generate p p > t t~
```

generates the highest powers in α_S: the LO_1 term
One can also specify the coupling orders (at cross section level!) by:

```
generate p p > t t~ aS==2 aEW==0
```

will generate the same diagrams:

```
MG5_aMC>generate p p > t t~
6 processes with 8 diagrams generated in 0.049 s
Total: 6 processes with 8 diagrams
MG5_aMC>generate p p > t t~ aS==2 aEW==0
change syntax aEW=0 to QED^2=0 to correspond to UFO model convention
change syntax aS=2 to QCD^2=4 to correspond to UFO model convention
6 processes with 8 diagrams generated in 0.030 s
Total: 6 processes with 8 diagrams
MG5_aMC>
```

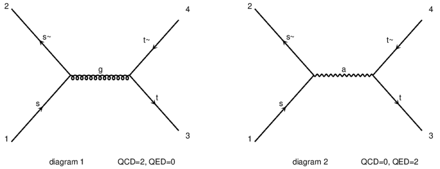
MG5_aMC syntax for complete-NLO computations

To generate the LO_2 ($\alpha_S \alpha$):

```
generate p p > t t~ aS==1 aEW==1
```

Note that $==$ sets the exact coupling orders, while $=$ is interpreted as $<=$!

Note that for this order, among others the following diagrams are generated because of their interface at cross section level:



MG5_aMC syntax for complete-NLO computations

For computing NLO QCD, we add the brackets:

```
generate p p > t t~ [QCD]
```

which generates LO_1 and NLO_1
For generating the other NLO terms, there are two options: either using the flag

```
generate p p > t t~ [QED]
```

which gives LO_1 and NLO_2.
Or using:

```
generate p p > t t~ aS=1 aEW=1 [QCD]
```

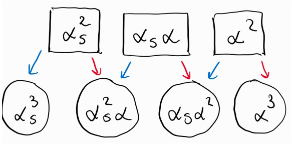
which generates LO_2 and NLO_2
One can also use both correction markers:

```
generate p p > t t~ aS=2 aEW=0 [QCD QED]
```

which generates LO_1 and $\text{NLO}_1+\text{NLO}_2$.

MG5_aMC syntax for complete-NLO computations

In simpler words:



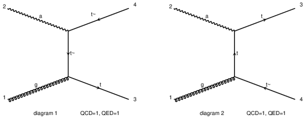
The [QCD] flag generates blue arrow corrections, [QED] flag generates red arrow, and aS=..., aEW= ... specify Born-level amplitude

Good to know but careful when using:

One can also set coupling order constraints on diagram level, using the syntax:

```
generate p p > t t~ QCD==1 QED==1
```

this will generate the two diagrams:



MG5_aMC syntax for complete-NLO computations

We generate as an example the process:

```
generate p p > t t~ aS=2 aEW=2 [QCD]
```

The coupling order constraints are stored in the

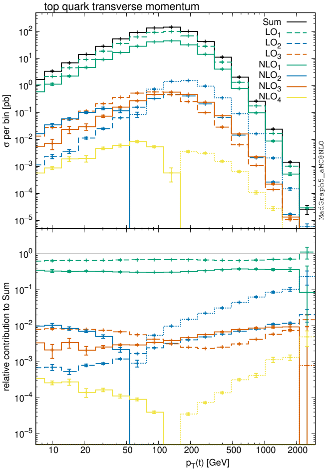
```
orders_tag_plot = QCD*1 + QED*100
```

which is stored in the run:

```
INFO: orders_tag_plot is computed as:         + QCD *        1          + QED *       100
 orders_tag_plot=             4  for QCD,QED, =            4 ,            0 ,
 orders_tag_plot=             6  for QCD,QED, =            6 ,            0 ,
 orders_tag_plot=           204  for QCD,QED, =            4 ,            2 ,
 orders_tag_plot=           402  for QCD,QED, =            2 ,            4 ,
```

MG5_aMC syntax for complete-NLO computations

- This can be used for plotting histograms for the various sub-contributions to complete-NLO and complete-LO
- Using HwU (Histograms with Uncertainties) internal MG5_aMC plotting routine, the orders can be simply obtained
- Have for example a look at /FixedOrderAnalysis/analysis_HwU_general.f
- **Example for** $pp \to t\bar{t}$**:** LO_1, LO_2, LO_3, NLO_1, NLO_2, NLO_3, NLO_4 separate contributions



MG5_aMC syntax for EW Sudakov

Note: only in MG5_aMC_v3_6_* version!

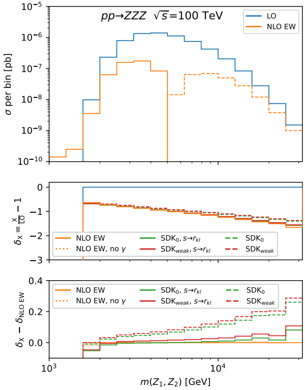
For including the EW Sudakov enhancement, one needs to

```
import model loop_qcd_qed_sm_Gmu_forSudakov
```

to include the correct model information.

For the process generation:

```
generate p p > t t~ [QCD] --ewsudakov
```



Matching NLO to parton showers

Today we have seen how to compute NLO cross sections! What if we want to combine these with the parton shower algorithm?

Recall LO merging: we had to introduce a cut-off t_{cut} which divided the hard matrix element region and the region which is generated by the parton shower

Can we do this in the case of NLO?

NO!

For NLO, we integrate over **full phase space** of the real emission, we cannot cut this!

Matching NLO to parton showers

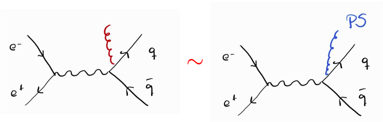
Recall the structure of the NLO integral with n-body and $(n+1)$-body events:

$$\sigma^{\mathrm{NLO}} = \int_n \mathrm{d}\sigma_B + \int_n \left(\mathrm{d}\sigma_V + \int_1 \mathrm{d}\sigma_C \right) + \int_{n+1} (\mathrm{d}\sigma_R - \mathrm{d}\sigma_C)$$

one option is to shower the separate events, starting from the different kinematics: schematically, we denote a shower algorithm with I_k^{MC} applied on a k-body event:

$$\sigma^{\mathrm{NLO}} = \left(\int_n \mathrm{d}\sigma_B + \int_n \left(\mathrm{d}\sigma_V + \int_1 \mathrm{d}\sigma_C \right) \right) I_n^{MC} + \int_{n+1} (\mathrm{d}\sigma_R - \mathrm{d}\sigma_C) \, I_{n+1}^{MC}$$

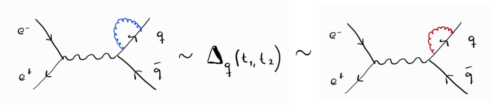
The issue is double-counting!



Matching NLO to parton showers

Double-counting in:

- Real emission and single PS emissions
- Virtual corrections and Sudakov form factor



Multiple algorithms/methods to do this consistently without the double counting:

- MC@NLO matching [Frixione, Webber, arXiv:hep-ph/0204244]
- POWHEG [Frixione, Nason, Oleari, arXiv:0709.2092]
- KrKNLO [Jadach et al., arXiv:1503.06849

Matching NLO to parton showers

The MC@NLO method is the matching scheme used in MG5_aMC@NLO
Introduces shower subtraction terms PS_{n+1} in the following way:

$$\underbrace{\left(\int_n d\sigma_B + \int_n \left(d\sigma_V + \int_1 d\sigma_C\right)\right) I_n^{MC} + \left(\int_{n+1} PS_{n+1} - d\sigma_C\right) I_n^{MC}}_{\text{S-events}}$$

$$+ \underbrace{\left(\int_{n+1} d\sigma_R - PS_{n+1}\right) I_{n+1}^{MC}}_{\text{H-events}}$$

which are respectively:

Born + subtracted virtual of NLO
virtual (Sudakov) of parton shower - subtraction part of NLO
Real part of NLO - *real part of shower*

and the shower subtraction term PS_{n+1} is defined as the parton shower algorithm to go from the n-body to the $(n+1)$-body phase space and **is shower-dependent**