Parton Showers
and their matching/merging with NLO computations

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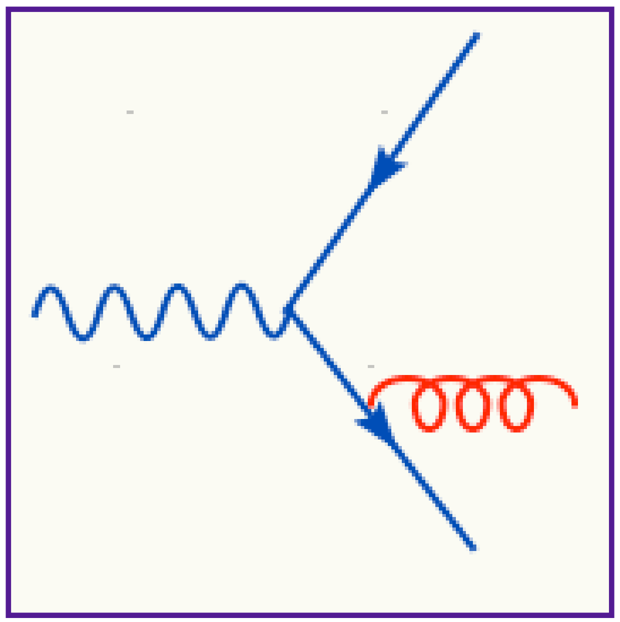

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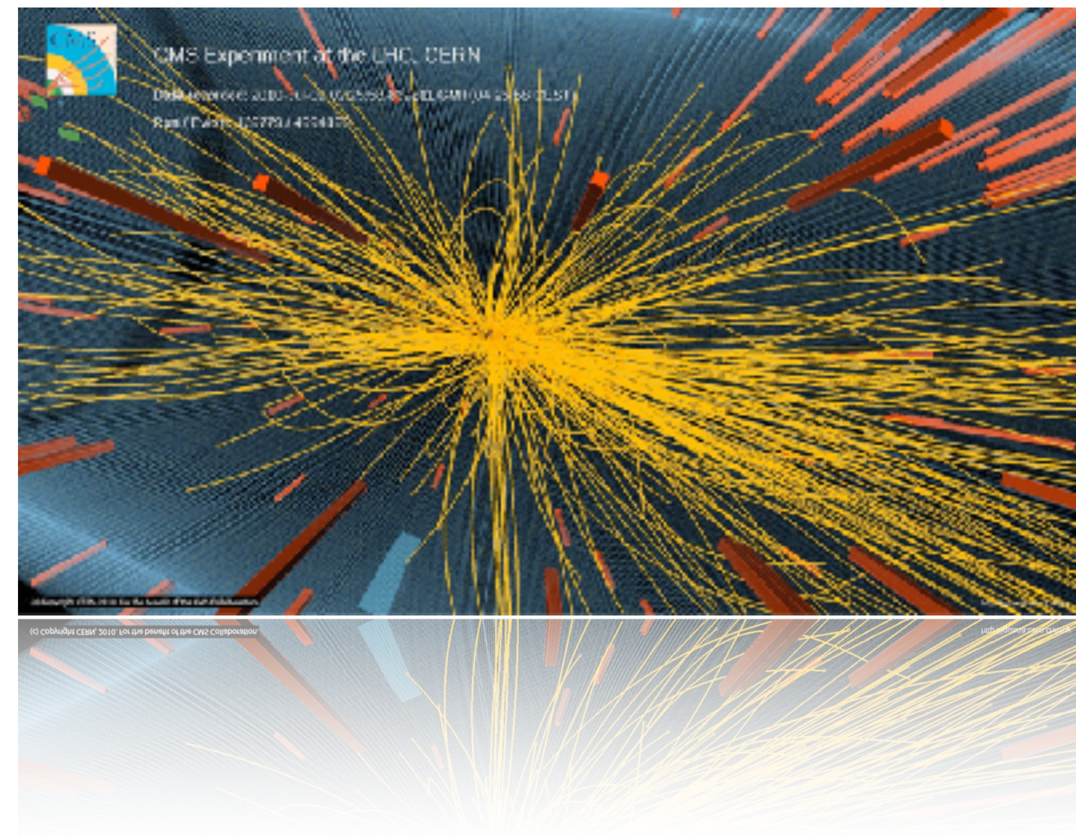

CMS Experiment at the LHC, CERN

An LHC collision, factorised

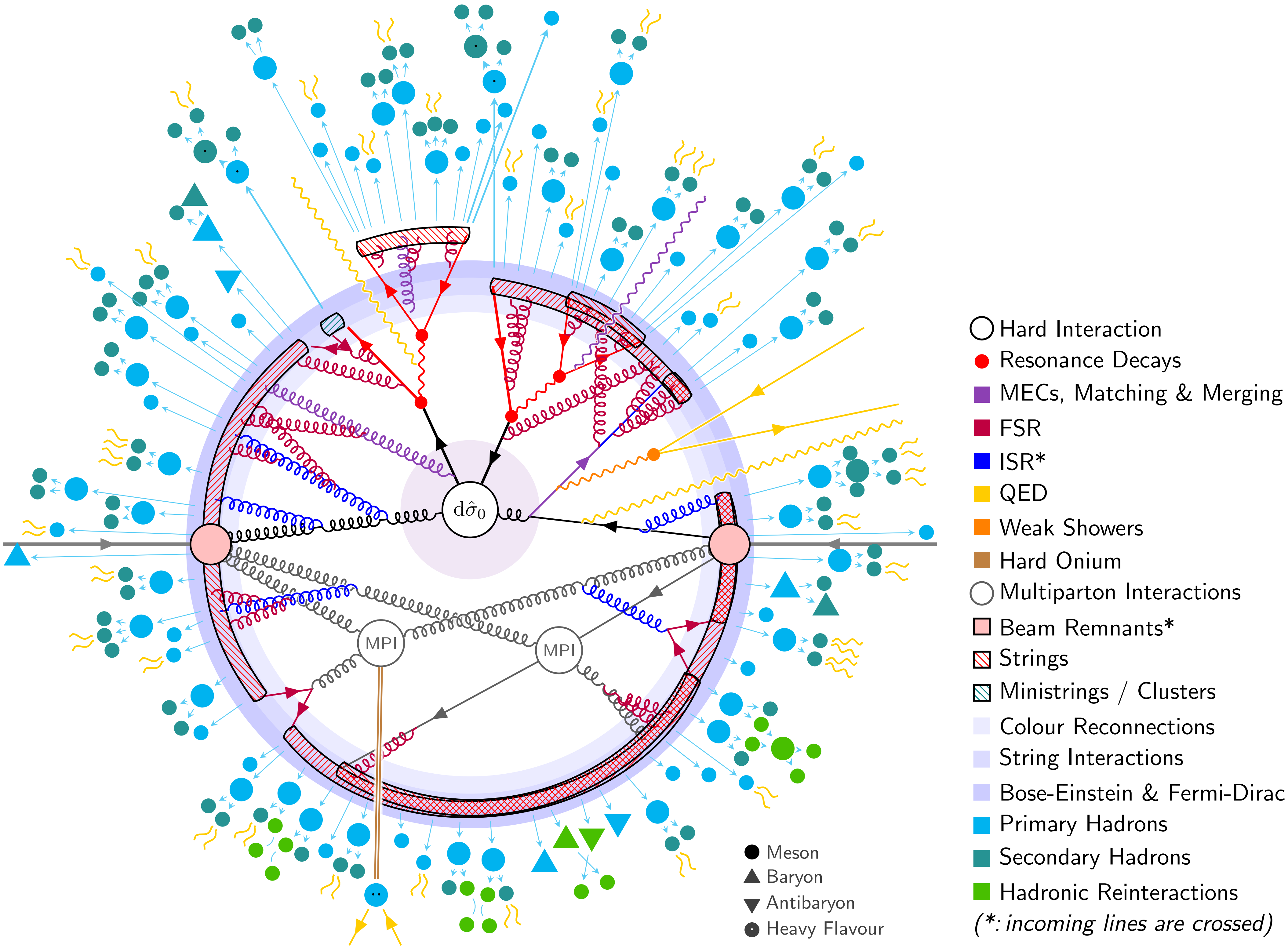
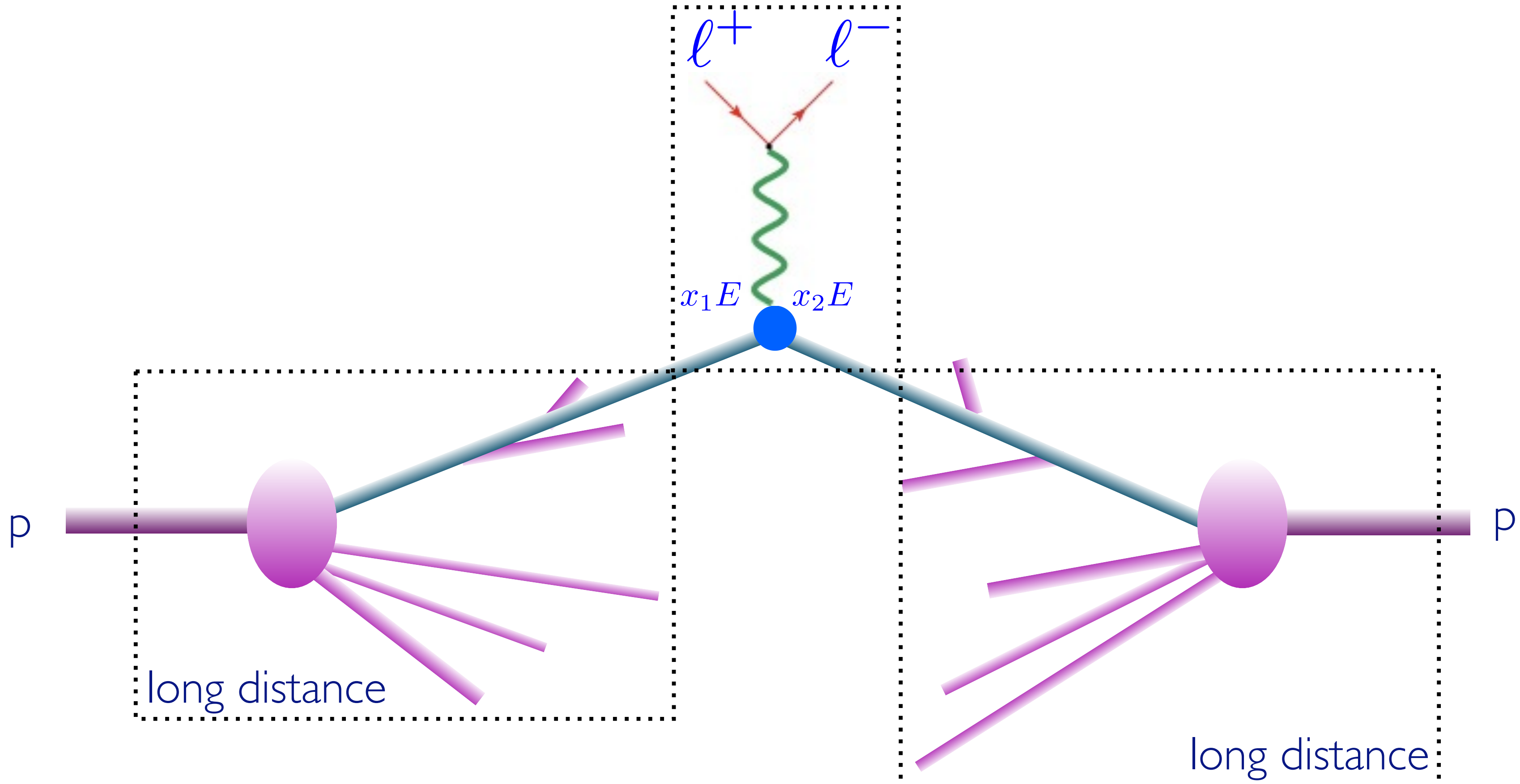


Figure taken from Bierlich et al., 2022 *(Pythia8.3 manual)*

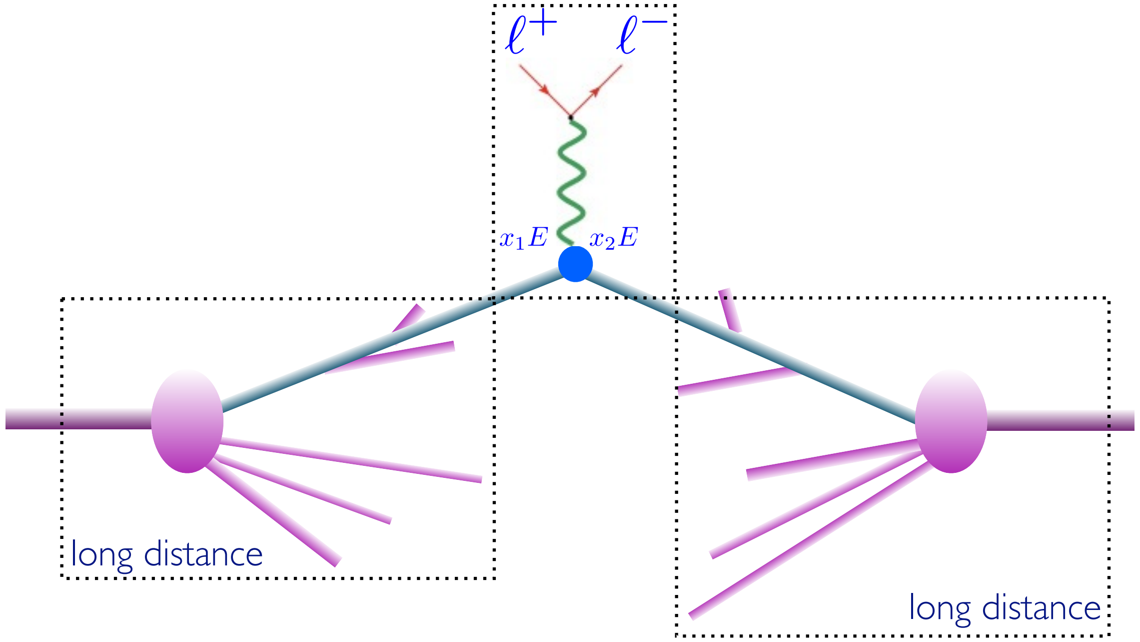
LHC master formula



$$\sigma_X = \sum_{a,b} \int_0^1 dx_1 dx_2 \, f_a(x_1, \mu_F^2) f_b(x_2, \mu_F^2) \times \hat{\sigma}_{ab\to X}(x_1, x_2, \alpha_S(\mu_R^2), \frac{Q^2}{\mu_F^2}, \frac{Q^2}{\mu_R^2})$$

Matrix elements in perturbation theory

- In matrix element calculations in perturbation theory (Olivier's lecture)
  - "Initial state QCD radiation" is included ("resummed") in the PDFs
  - Similarly "final state QCD radiation" is included through the parton-jet duality
- Hence… all is already there! What to do…?
  - "Undo" this resummation and make it explicit



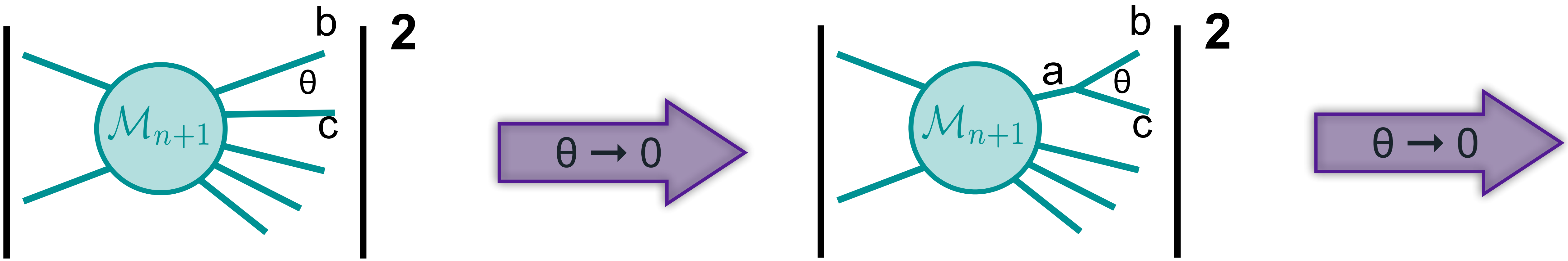
Why do this?

- Effects are already included (resummed) in fixed-order perturbation theory
  - for many *inclusive* observables including shower should not influence the results
  - but one would miss an extremely rich variety of observables which may play important roles in experimental analyses.

- When there are large scale differences entering the observables, fixed-order perturbation theory breaks down!
  - this does NOT mean that observable is useless/unimportant: it is just that one is not using the right tools to describe it.
  - It is better to try and find a way to reorganise the computation in order to take into account emissions close to the singular regions of the phase space, to all orders in perturbation theory.

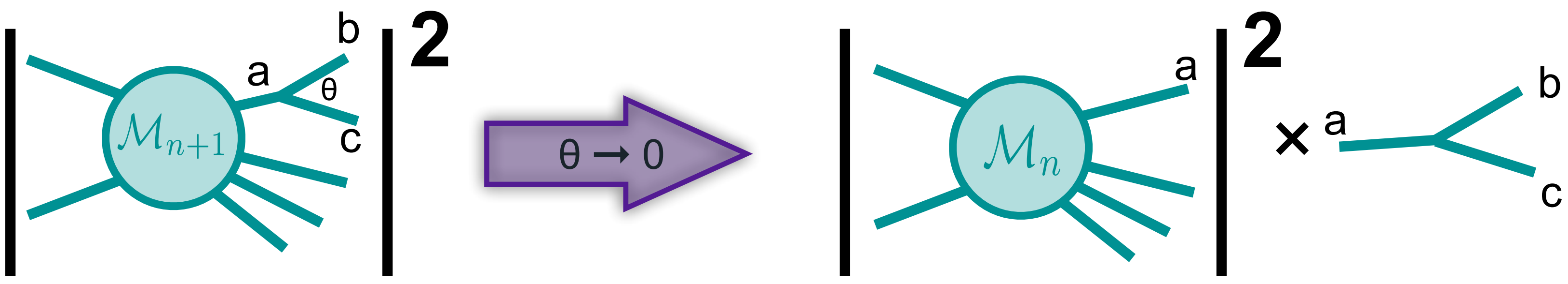
- "We want to simulate the collisions, hence we want to simulate also the creation of the hadrons, for which we need parton showering"

Collinear factorisation



- Consider a process for which two particles are separated by a small angle θ
- In the limit of θ → 0, the contribution is coming from a single parent particle going on shell: therefore its branching is related to time scales which are very long with respect to the hard subprocess
- The inclusion of such a branching cannot change the picture set up by the hard process: the whole emission process must be writable in this limit as the simpler one times a branching probability

Collinear factorisation



- The process factorises in the collinear limit. This procedure is universal

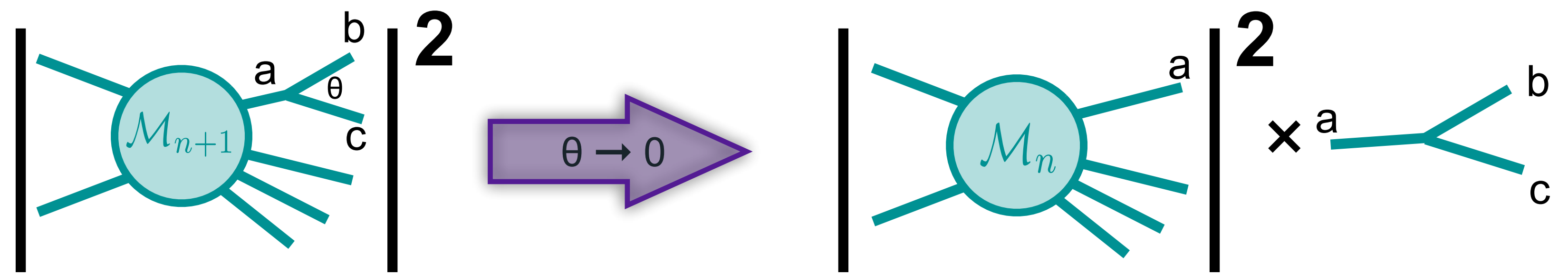
$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \simeq |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

- Notice that what has been roughly called ‘branching probability’ is actually a singular factor, so one will need to make sense of this definition.
- At the leading contribution to the (n+1)-body cross section the DGLAP splitting kernels are defined as:

$$P_{g\to qq}(z) = T_R\left[z^2 + (1-z)^2\right], \qquad P_{g\to gg}(z) = C_A\left[z(1-z) + \frac{z}{1-z} + \frac{1-z}{z}\right],$$

$$P_{q\to qg}(z) = C_F\left[\frac{1+z^2}{1-z}\right], \qquad P_{q\to gq}(z) = C_F\left[\frac{1+(1-z)^2}{z}\right].$$

Collinear factorisation



- The process factorises in the collinear limit. This procedure is universal

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \ \simeq \ |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_s}{2\pi} P_{a\to bc}(z)$$

- t can be called the 'evolution variable' (will become clearer later): it can be the virtuality m^2 of particle a, or its p_T^2, or $E^2\theta^2$ …
  - It represents the hardness of the branching and tends to 0 in the collinear limit.

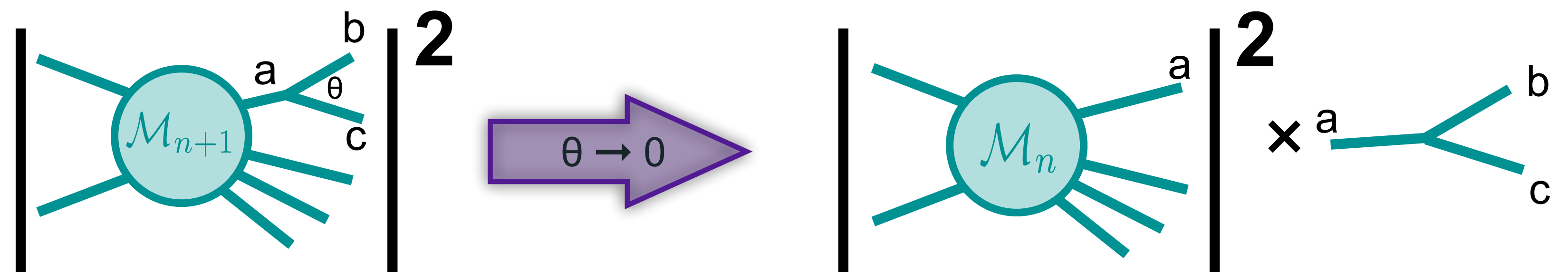
$$m^2 \simeq z(1-z)\theta^2 E_a^2$$

$$p_T^2 \simeq z m^2$$

- Indeed in the collinear limit one has: so that the factorisation takes place for all these definitions:

$$d\theta^2/\theta^2 \ = \ dm^2/m^2 \ = \ dp_T^2/p_T^2$$

Collinear factorisation

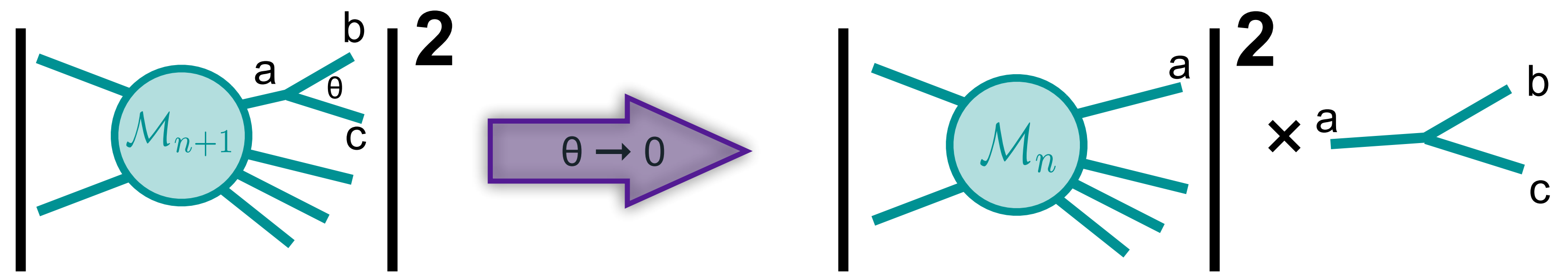


- The process factorises in the collinear limit. This procedure is universal

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \simeq |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_s}{2\pi} P_{a\to bc}(z)$$

- z is the "energy variable": it is defined to be the energy fraction taken by parton b from parton a
  - It represents the energy sharing between b and c and tends to 1 in the soft limit (parton c going soft)
- ϕ is the azimuthal angle. It can be chosen to be the angle between the polarisation of a and the plane of the branching

Collinear factorisation

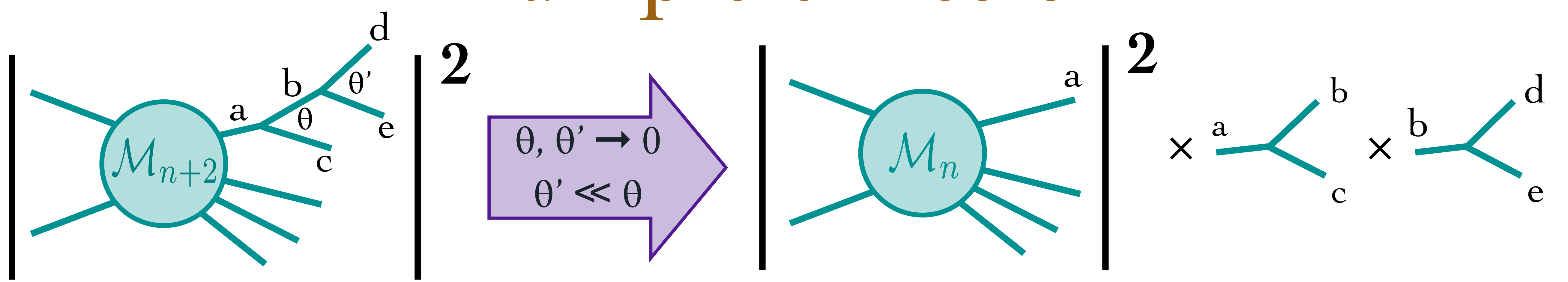


- The process factorises in the collinear limit. This procedure is universal

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \simeq |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_s}{2\pi} P_{a\to bc}(z)$$

- This is an amplitude squared: naively one would maybe expect $1/t^2$ dependence. Why is the square not there?
  - It's due to angular-momentum conservation.
    E.g., take the splitting $q \longrightarrow qg$: helicity is conserved for the quarks, so the final state spin differs by one unity with respect to the initial one. The scattering happens in a p-wave (orbital angular momentum equal to one), so there is a suppression factor as $t \longrightarrow 0$.
    - Indeed, a factor $1/t$ is always cancelled in an explicit computation

Multiple emission

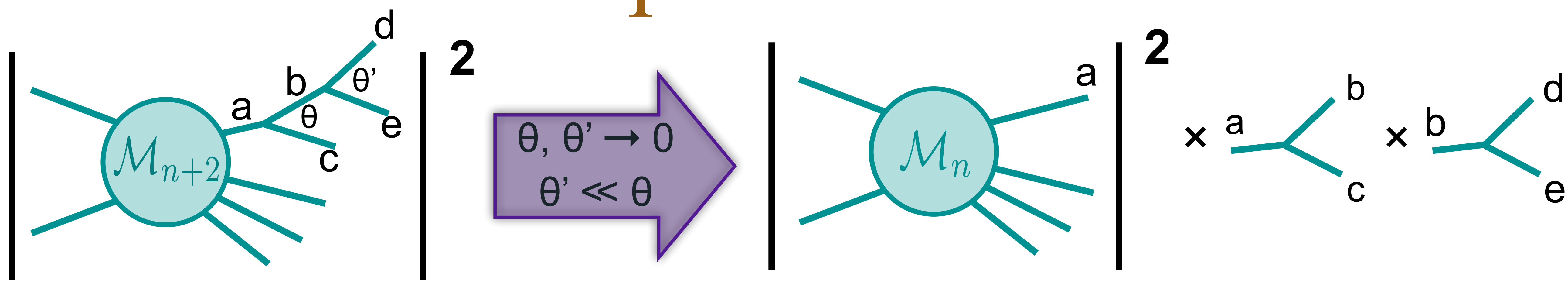


- Now consider $\mathcal{M}_{n+2}$ as the new core process and use the recipe we used for the first emission in order to get the dominant contribution to the (n+2)-body cross section: add a new branching at angle much smaller than the previous one:

$$|\mathcal{M}_{n+2}|^2 d\Phi_{n+2} \simeq |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_s}{2\pi} P_{a\to bc}(z) \times \frac{dt'}{t'} dz' \frac{d\phi'}{2\pi} \frac{\alpha_s}{2\pi} P_{b\to de}(z')$$

- This can be done for an arbitrary number of emissions. The recipe to get the leading collinear singularity is thus cast in the form of an iterative sequence of emissions whose probability does not depend on the past history of the system: a 'Markov chain'.

Multiple emissions



- The dominant contribution comes from the region where the subsequently emitted partons satisfy the strong ordering requirement: θ ≫ θ' ≫ θ''...
  For the rate for multiple emission we get

$$\sigma_{n+k} \propto \alpha_S^k \int_{Q_0^2}^{Q^2} \frac{dt}{t} \int_{Q_0^2}^{t} \frac{dt'}{t'} \cdots \int_{Q_0^2}^{t^{(k-2)}} \frac{dt^{(k-1)}}{t^{(k-1)}} \propto \sigma_n \left(\frac{\alpha_S}{2\pi}\right)^k \log^k(Q^2/Q_0^2)$$

  where Q is a typical hard scale and Q_0 is a small infrared cutoff that separates perturbative from non perturbative regimes.
- Each power of α_S comes with a logarithm. The logarithm can easily be large, and therefore we see a breakdown of perturbation theory

Absence of interference

- The collinear factorisation picture gives a branching sequence for a given leg starting from the hard subprocess all the way down to the non-perturbative region.
- Suppose you want to describe two such histories from two different legs:
  - these two legs are treated in a completely uncorrelated way. And even within the same history, subsequent emissions are uncorrelated.
- The collinear picture completely misses the possible interference effects between the various legs
  - the extreme simplicity comes with the price of quantum inaccuracy.
- **Smart choices improve upon this: soft enhancement (which is purely an interference contribution) can be included. For this, the evolution variable must be related to the angle of the emission**
- Nevertheless, the collinear picture captures the leading contributions: it gives an excellent description of an arbitrary number of (collinear) emissions:
  - it is a “resummed computation” and
  - it bridges the gap between fixed-order perturbation theory and the non-perturbative hadronisation.

Emission probability & Sudakov form factor

- The differential probability for the branching $a \rightarrow bc$ between scales t and t+dt knowing that no emission occurred before:

$$dp(t) = \sum_{bc} \frac{dt}{t} \int dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\rightarrow bc}(z)$$

- The probability that a parton does NOT split between the scales t and t+dt is given by 1-dp(t)

- Probability that particle a does not emit between scales Q^2 and t

$$\Delta(Q^2, t) = \prod_k \left[1 - \sum_{bc} \frac{dt_k}{t_k} \int dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\rightarrow bc}(z)\right] =$$

$$\exp\left[-\sum_{bc} \int_t^{Q^2} \frac{dt'}{t'} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\rightarrow bc}(z)\right] = \exp\left[-\int_t^{Q^2} dp(t')\right]$$

$\Delta(Q^2,t)$ is the Sudakov form factor

Parton shower

- The Sudakov form factor is the heart of the parton shower. It gives the probability that a parton does not branch between two scales
  *Initial state shower also requires PDF contributions
- Using this no-emission probability one can generate the branching tree of a parton
- Define dP_k as the probability for k ordered splittings from leg a at given scales

$$\begin{aligned} dP_1(t_1) &= \Delta(Q^2, t_1)\; dp(t_1)\Delta(t_1, Q_0^2), \\ dP_2(t_1, t_2) &= \Delta(Q^2, t_1)\; dp(t_1)\; \Delta(t_1, t_2)\; dp(t_2)\; \Delta(t_2, Q_0^2)\Theta(t_1 - t_2), \\ \ldots &= \ldots \\ dP_k(t_1, \ldots, t_k) &= \Delta(Q^2, Q_0^2) \prod_{l=1}^{k} dp(t_l)\Theta(t_{l-1} - t_l) \end{aligned}$$

- Q_0^2 is the hadronisation scale (~1 GeV^2). Below this scale we do not trust the perturbative description for parton splitting anymore
- This is what is implemented in a parton shower, taking the scales for the splitting t_i randomly (but weighted according to the no-emission probability)

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Unitarity

$$dP_k(t_1, ..., t_k) \quad = \quad \Delta(Q^2, Q_0^2) \prod_{l=1}^{k} dp(t_l)\Theta(t_{l-1} - t_l)$$

- The parton shower has to be unitary (the sum over all branching trees should be 1). We can explicitly check this by integrating the probability for k splittings

$$P_k \equiv \int dP_k(t_1, ..., t_k) = \Delta(Q^2, Q_0^2) \frac{1}{k!} \left[\int_{Q_0^2}^{Q^2} dp(t) \right]^k , \quad \forall k = 0, 1, ...$$

- Summing over all number of emissions

$$\sum_{k=0}^{\infty} P_k = \Delta(Q^2, Q_0^2) \sum_{k=0}^{\infty} \frac{1}{k!} \left[\int_{Q_0^2}^{Q^2} dp(t) \right]^k = \Delta(Q^2, Q_0^2) \exp \left[\int_{Q_0^2}^{Q^2} dp(t) \right] = 1$$

- Hence, the total probability is conserved

Cancellation of singularities

- We have shown that the shower is unitary. However, how are the IR divergences cancelled explicitly? Let's show this for the first emission: Consider the contributions from (exactly) 0 and 1 emissions from leg a:

$$\frac{d\sigma}{\sigma_n} = \Delta(Q^2, Q_0^2) + \Delta(Q^2, Q_0^2) \sum_{bc} dz \frac{dt}{t} \frac{d\phi}{2\pi} \frac{\alpha_\mathrm{S}}{2\pi} P_{a\to bc}(z)$$

- Expanding to first order in α_s gives

$$\frac{d\sigma}{\sigma_n} \simeq 1 - \sum_{bc} \int_{Q_0^2}^{Q^2} \frac{dt'}{t'} dz \frac{d\phi}{2\pi} \frac{\alpha_\mathrm{S}}{2\pi} P_{a\to bc}(z) + \sum_{bc} dz \frac{dt}{t} \frac{d\phi}{2\pi} \frac{\alpha_\mathrm{S}}{2\pi} P_{a\to bc}(z)$$

- Same structure of the two latter terms, with opposite signs: cancellation of divergences between the approximate virtual and approximate real emission cross sections.

- The probabilistic interpretation of the shower ensures that infrared divergences will cancel for each emission.

Argument of α_S

- Each choice of argument for α_S is equally acceptable at the leading-logarithmic accuracy. However, there is a choice that allows one to resum certain classes of subleading logarithms.
- The higher order corrections to the partons splittings imply that the DGLAP splitting kernels should be modified: $P_{a\to bc}(z) \longrightarrow P_{a\to bc}(z) + \alpha_S P'_{a\to bc}(z)$
- For $g \to gg$ branchings $P'_{a\to bc}(z)$ diverges as $-b_0 \log[z(1-z)]\ P_{a\to bc}(z)$ (just z or $1-z$ if quark is present)
- Recall the one-loop running of the strong coupling

$$\alpha_S(Q^2) = \frac{\alpha_S(\mu^2)}{1+\alpha_S(\mu^2) b_0 \log\frac{Q^2}{\mu^2}} \sim \alpha_S(\mu^2)\left(1-\alpha_S(\mu^2) b_0 \log\frac{Q^2}{\mu^2}\right)$$

- We can therefore include the $P'(z)$ terms by choosing $p_T^2 \sim z(1-z)Q^2$ as argument of α_S:

$$\begin{aligned}\alpha_S(Q^2)\left(P_{a\to bc}(z)+\alpha_S(Q^2)P'_{a\to bc}\right) &= \alpha_S(Q^2)\left(1-\alpha_S(Q^2) b \log z(1-z)\right)P_{a\to bc}(z)\\ &\sim \alpha_S(z(1-z)Q^2)P_{a\to bc}(z)\end{aligned}$$

Choice of evolution parameter

$$\Delta(Q^2,t) = \exp\left[-\sum_{bc}\int_t^{Q^2}\frac{dt'}{t'}dz\frac{d\phi}{2\pi}\frac{\alpha_S}{2\pi}P_{a\to bc}(z)\right]$$

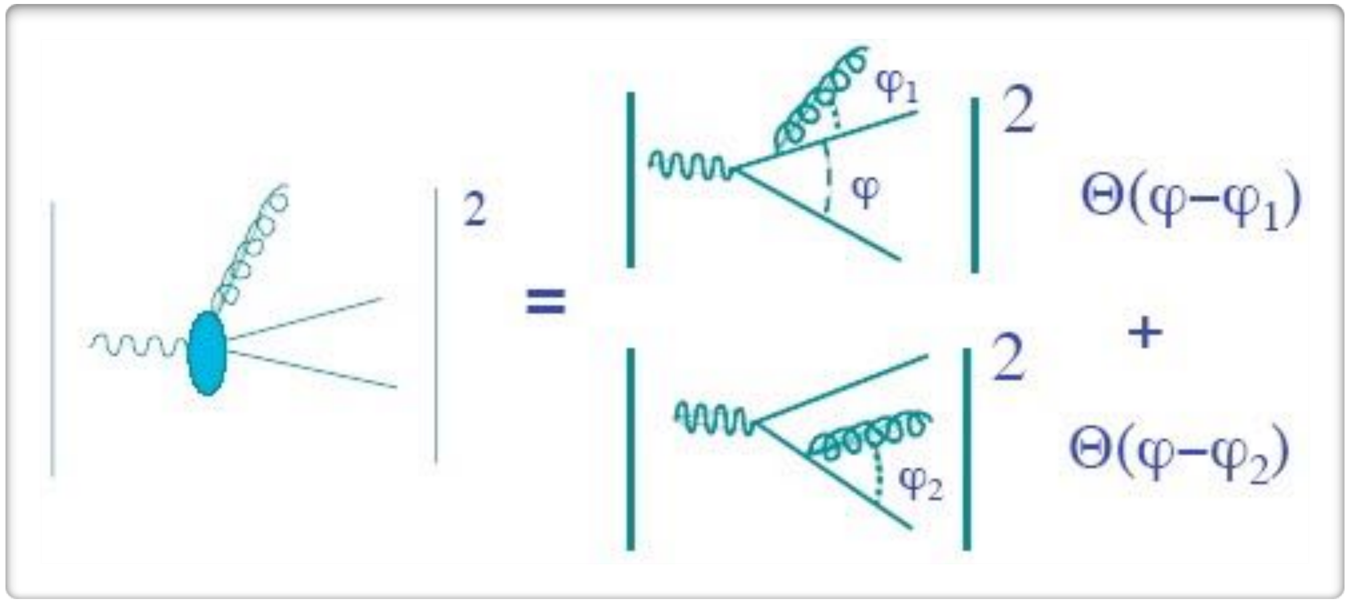
- There is a lot of freedom in the choice of evolution parameter t. It can be the virtuality m^2 of particle a or its p_T^2 or $E^2\theta^2$ ... For the collinear limit they are all equivalent

$$d\theta^2/\theta^2 \ = \ dm^2/m^2 \ = \ dp_T^2/p_T^2$$

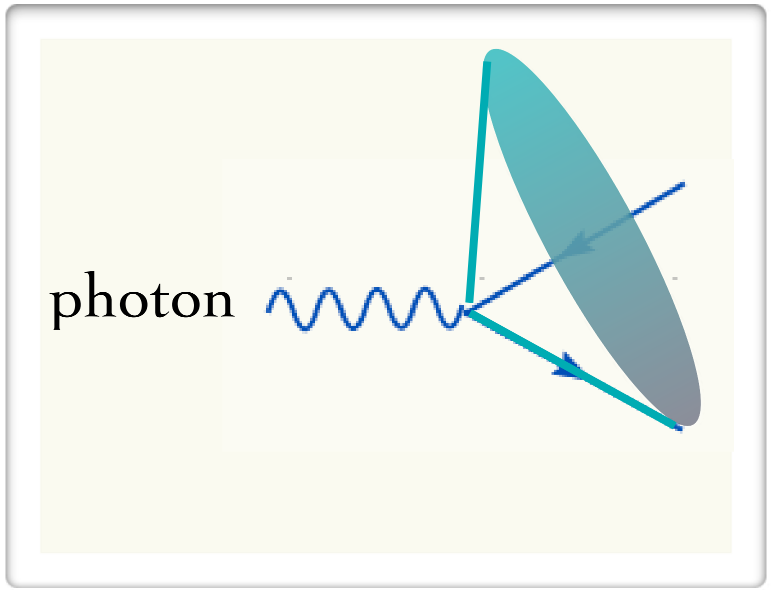
- However, in the soft limit ($z \longrightarrow 1$) they behave differently
- Can we chose it such that we get the correct soft limit?

YES! It should be (proportional to) the angle θ

Angular ordering



- Radiation inside cones around the original partons is allowed (and described by the eikonal approximation), outside the cones it is zero (after averaging over the azimuthal angle)



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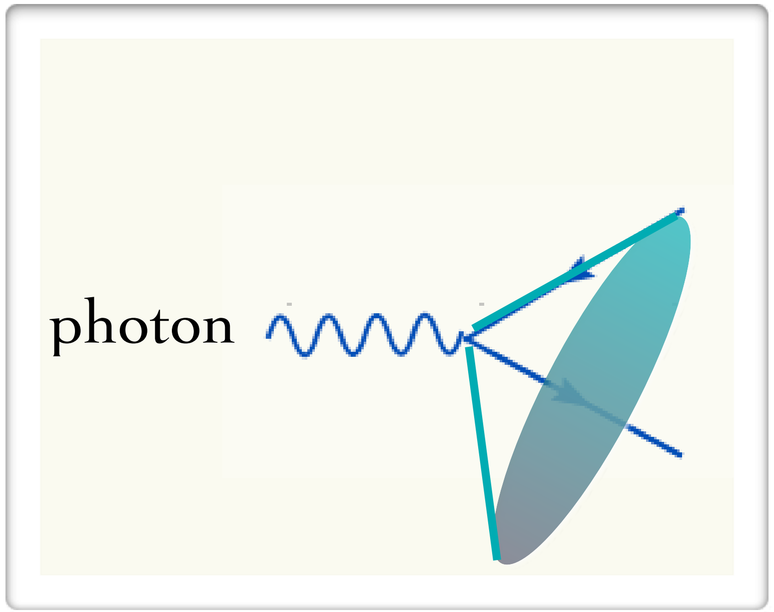
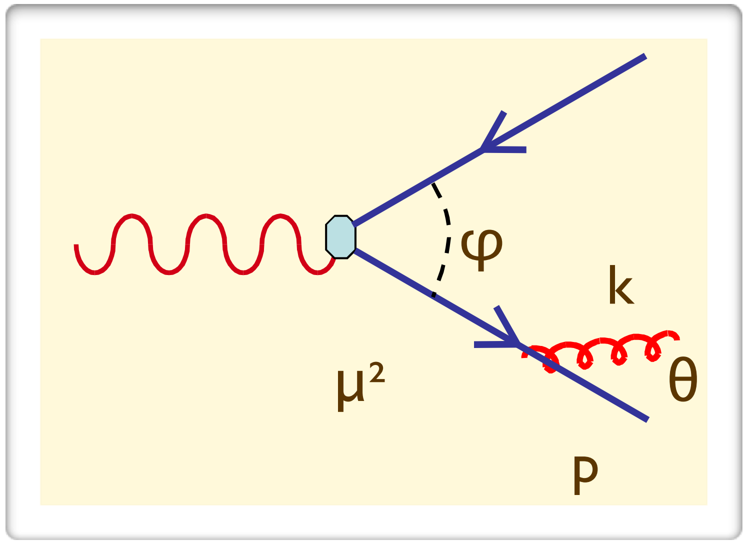


Intuitive explanation



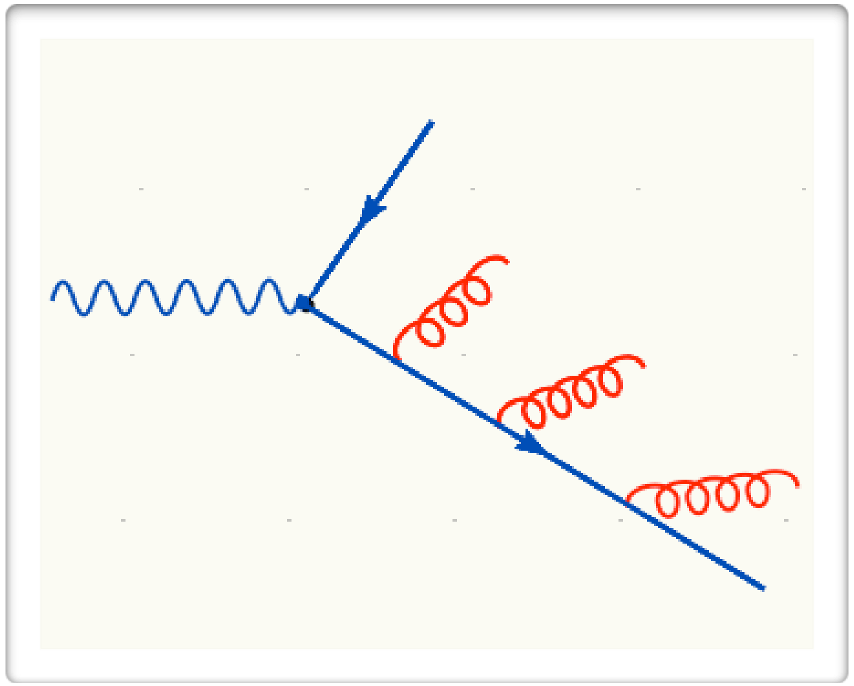
- Lifetime of the virtual intermediate state:
  $\tau < \gamma/\mu = E/\mu^2 = 1/(k_0\theta^2) = 1/(k_\perp\theta)$
- Distance between q and qbar after τ:
  $d = \varphi\tau = (\varphi/\theta)\ 1/k_\perp$

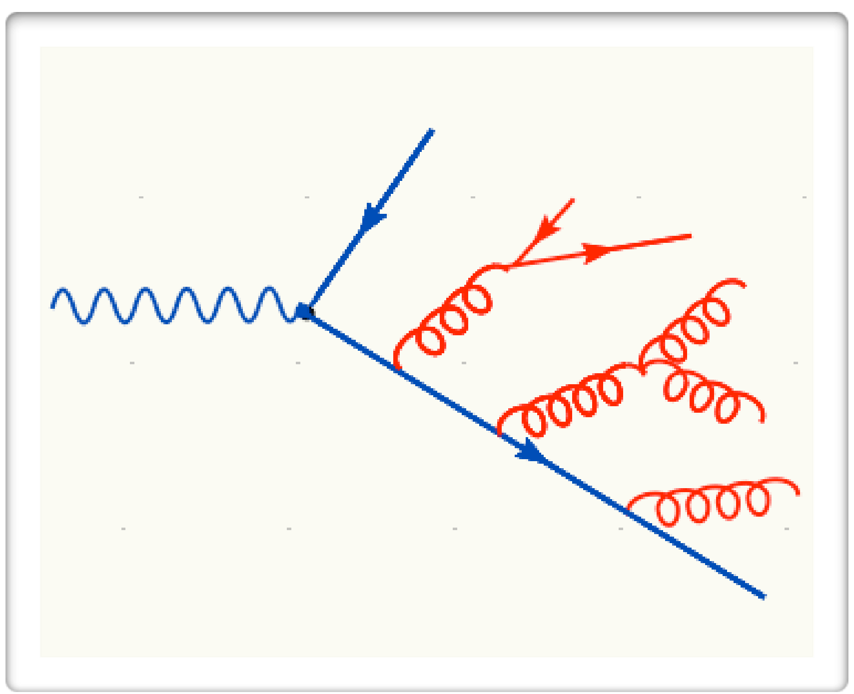
$\mu^2 = (p+k)^2 = 2E\ k_0\ (1\text{-}\cos\theta)$
$\sim E\ k_0\ \theta^2 \sim E\ k_\perp\ \theta$

If the transverse wavelength of the emitted gluon is longer than the separation between q and qbar, the gluon emission is suppressed, because the q qbar system will appear as colour neutral (i.e. dipole-like emission, suppressed)

Therefore $d>1/k_\perp$, which implies $\theta < \varphi$

Angular ordering





- The construction can be iterated to the next emission, with the result that the emission angles keep getting smaller and smaller.
- One can generalise it to a generic parton of colour charge Q_k splitting into two partons i and j, $Q_k=Q_i+Q_j$. The result is that inside the cones i and j emit as independent charges, and outside their angular-ordered cones the emission is coherent and can be treated as if it was directly from colour charge Q_k.
- Angular ordering is automatically satisfied in θ ordered showers! (and straight-forward to account for in p_T ordered showers)

Angular ordering

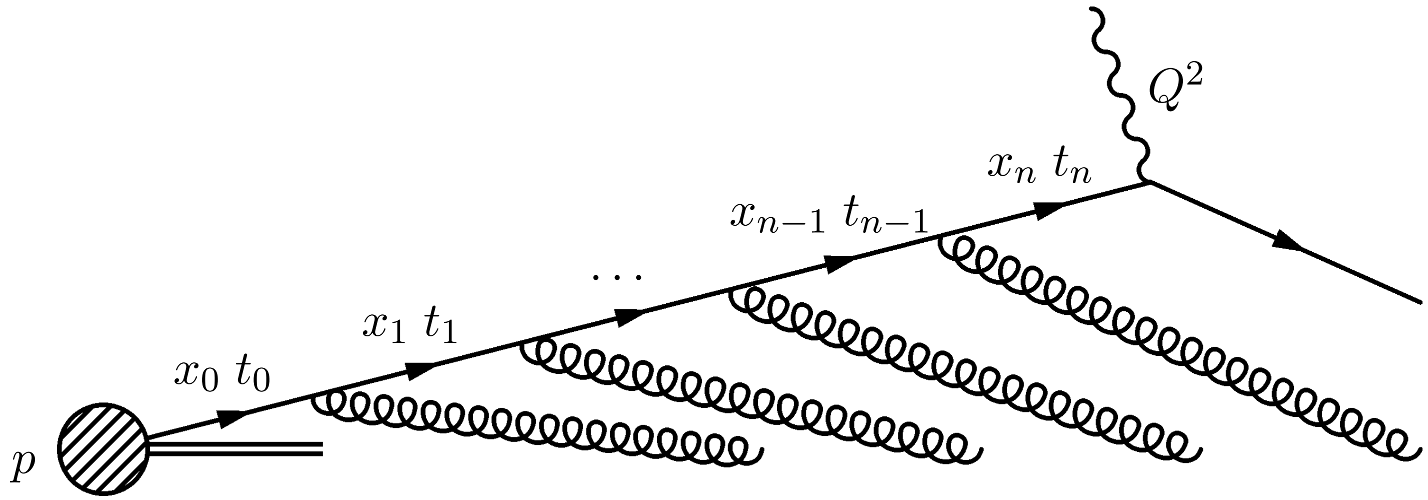
Angular ordering is:

1. A quantum effect coming from the interference of different Feynman diagrams.
2. Nevertheless it can be expressed in “a classical fashion” (square of an amplitude is equal to the sum of the squares of two special “amplitudes”). The classical limit is the dipole-radiation.
3. It is not an exclusive property of QCD (i.e., it is also present in QED) but in QCD produces very non-trivial effects, depending on how particles are colour connected.

Initial-state parton splittings

- So far, we have looked at final-state (time-like) splittings
- For initial state, the splitting functions are the same
- However, there is another ingredient:
  the parton density (or distribution) functions (PDFs)
  - ➡ Naively: Probability to find a given parton in a hadron at a given momentum fraction $x = p_z/P_z$ and scale t
- How do the PDFs evolve with increasing t?

Initial-state parton splittings



- Start with a quark PDF $f_0(x)$ at scale t_0. After considering a single parton emission, the probability to find the quark at virtuality $t > t_0$ is

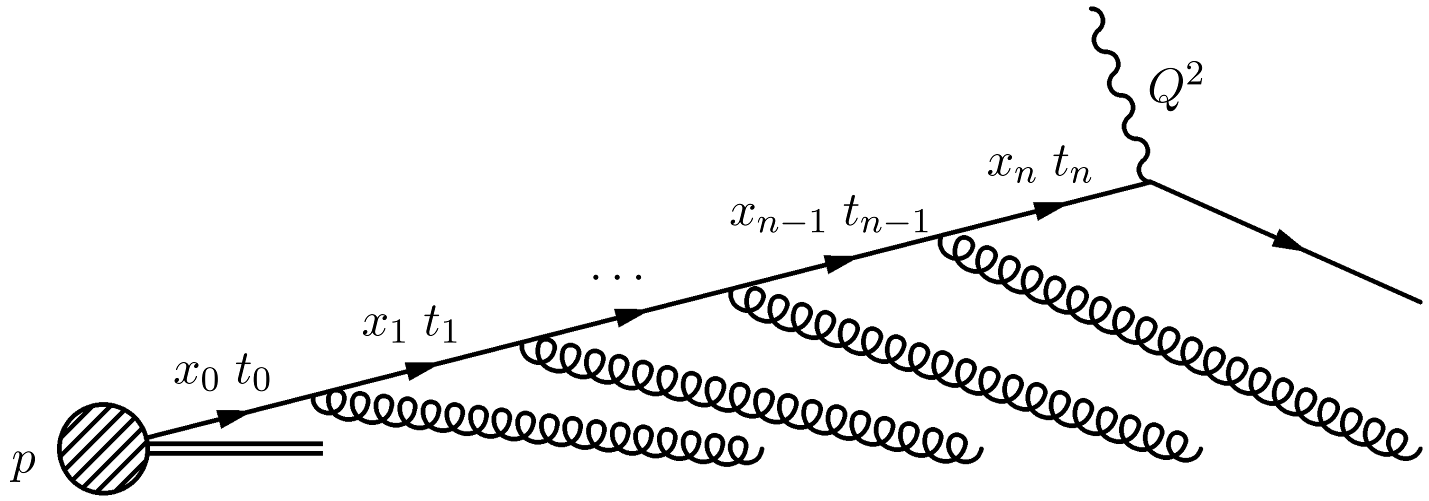
$$f(x,t) = f_0(x) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P(z) f_0\left(\frac{x}{z}\right)$$

- After a second emission, we have

$$f(x,t) = f_0(x) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P(z) \Big\{ f_0\left(\frac{x}{z}\right) + \int_{t_0}^{t'} \frac{dt''}{t''} \frac{\alpha_s}{2\pi} \int_{x/z}^1 \frac{dz'}{z'} P(z') f_0\left(\frac{x}{zz'}\right) \Big\}$$

$f(x/z, t')$

The DGLAP equation



- So for multiple parton splittings, we arrive at an integral-differential equation:

$$t\frac{\partial}{\partial t} f_i(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_s}{2\pi} P_{ij}(z) f_j\left(\frac{x}{z}, t\right)$$

- This is the famous DGLAP equation (where we have taken into account the multiple parton species i, j). The boundary condition for the equation is the initial PDFs $f_{i0}(x)$ at a starting scale t_0 (around 2 GeV).
- These starting PDFs are fitted to experimental data.

Initial-state parton showers

- To simulate parton radiation from the initial state, we start with the hard scattering, and then "devolve" the DGLAP evolution to get back to the original hadron: backwards evolution!
  - i.e. we undo the analytic resummation and replace it with explicit partons (e.g. in Drell-Yan this gives non-zero p_T to the vector boson)
- In backwards evolution, the Sudakovs include also the PDFs -- this follows from the DGLAP equation and ensures conservation of probability:

$$\Delta_{Ii}(x, t_1, t_2) = \exp\left\{ - \int_{t_1}^{t_2} dt' \sum_j \int_x^1 \frac{dx'}{x'} \frac{\alpha_s(t')}{2\pi} P_{ij}\left(\frac{x}{x'}\right) \frac{f_i(x', t')}{f_j(x, t')} \right\}$$

  This represents the probability that parton i will stay at the same x (no splittings) when evolving from t_1 to t_2.

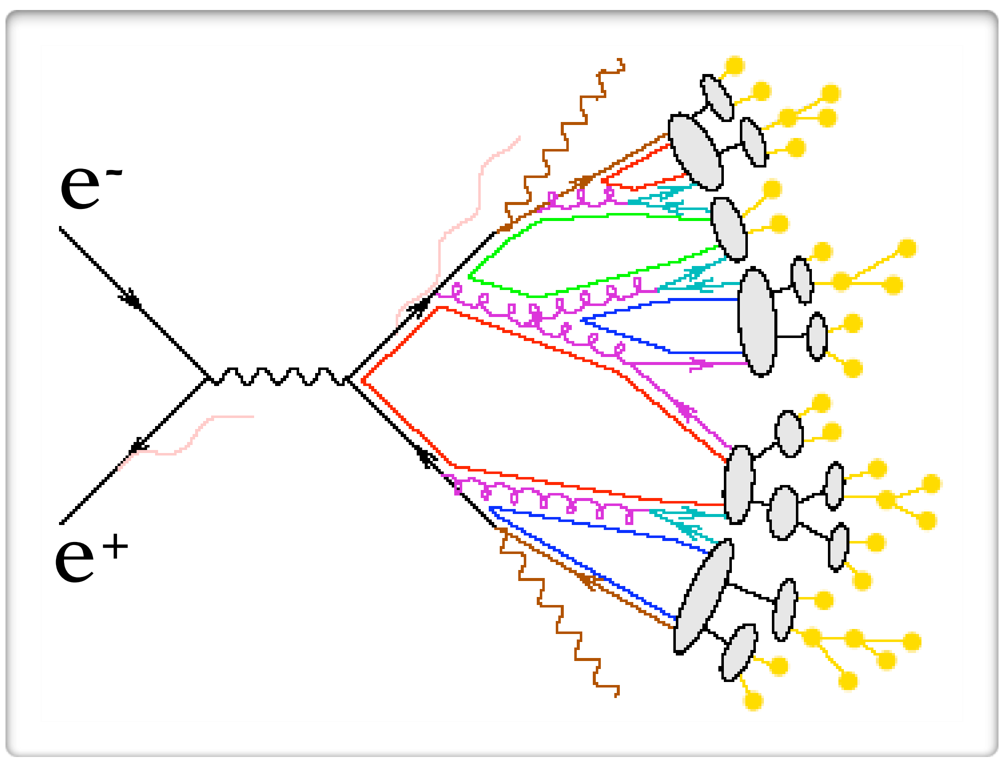
- The shower simulation is now done as in a final state shower

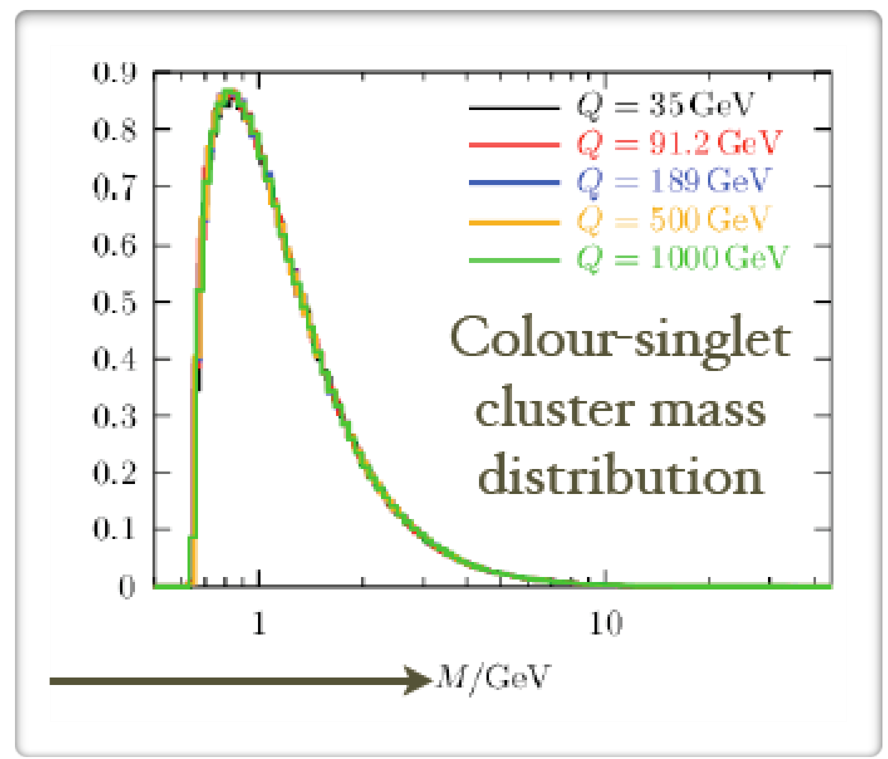
Hadronisation

- The shower stops if all partons are characterised by a scale at the IR cut-off: $Q_0 \sim 1$ GeV
- Physically, we observe hadrons, not (coloured) partons
- We need a non-perturbative model in passing from partons to colourless hadrons
- There are two models, based on physical and phenomenological considerations

Cluster model

The structure of the perturbative evolution including angular ordering, leads naturally to the clustering in phase-space of colour-singlet parton pairs (pre-confinement). Long-range correlations are strongly suppressed. Hadronisation will only act locally, on low-mass colour singlet clusters.





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String model

From lattice QCD one sees that the colour confinement potential of a quark-antiquark grows linearly with their distance: *V(r)* ~ *kr*, with *k* ~ 0.2 GeV, This is modelled with a string with uniform tension (energy per unit length) k that gets stretched between the qq̄ pair.

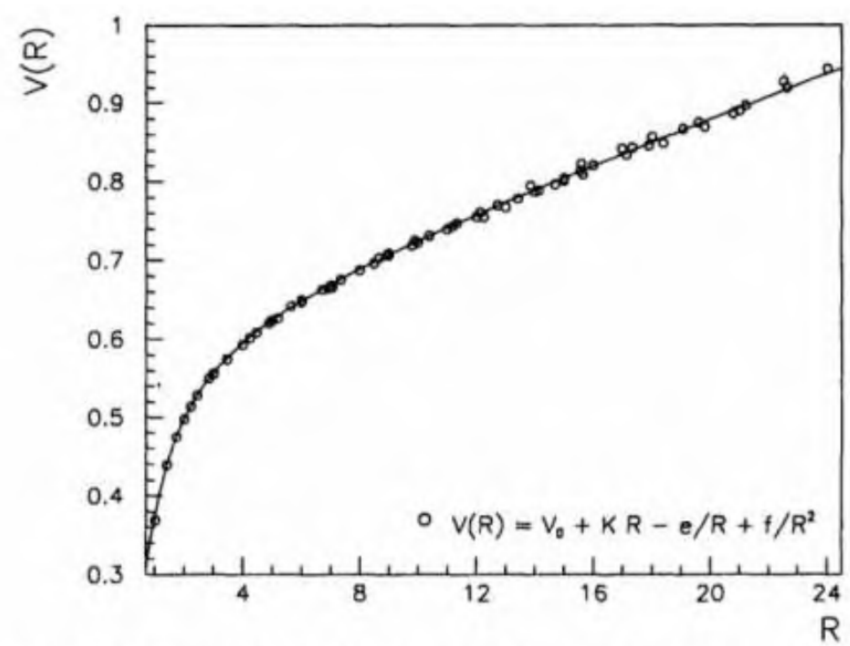
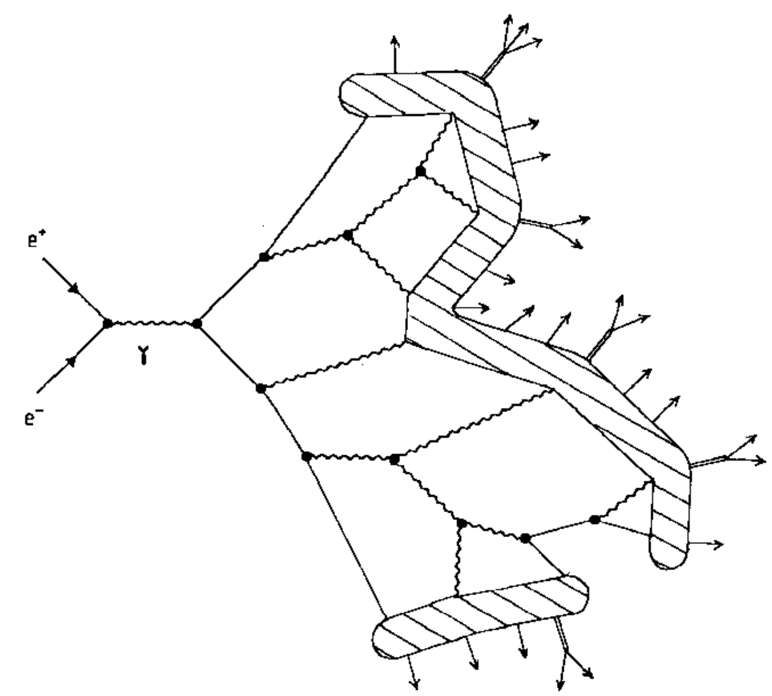
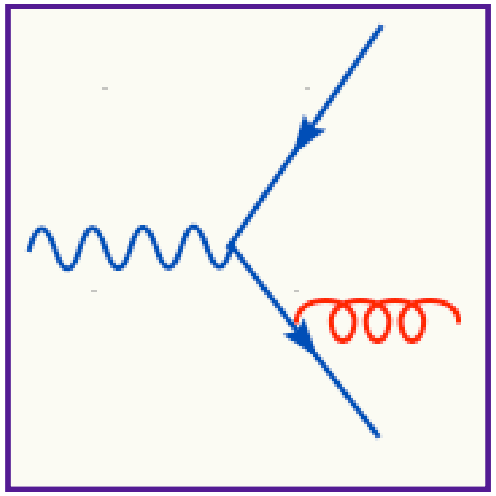


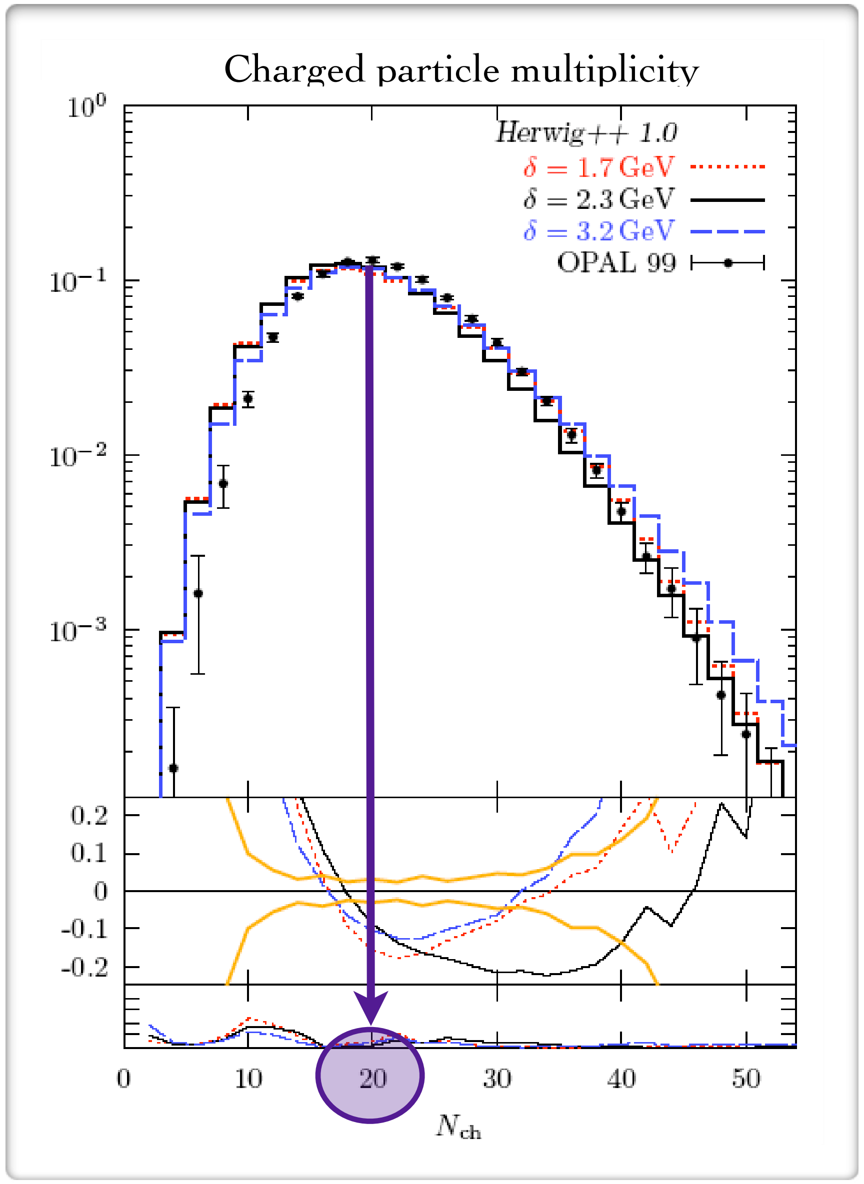
Fig. 2.9. QCD potential *vs.* *R* (in lattice units) from lattice QCD. Figure from ref. [23].



When quark-antiquarks are too far apart, it becomes energetically more favourable to break the string by creating a new qq̄ pair in the middle.

Exclusive observable





A parton shower program associates one of the possible histories (and pre-histories in case of pp collisions) of an hard event in an explicit and fully detailed way, such that the sum of the probabilities of all possible histories is unity.

Parton Shower MC event generators

A parton shower program associates one of the possible histories (and pre-histories in case of pp) of an hard event in an explicit and fully detailed way, such that the sum of the probabilities of all possible histories is unity.

- General-purpose tools
- Always the first experimental choice
- Complete exclusive description of the events: hard scattering, showering & hadronisation (and underlying event)
- Reliable and well-tuned tools
- Significant and intense progress in the development of new showering algorithms with the final aim to go at NLO in QCD

Pythia, Sherpa, Herwig, …

- Significant differences between shower implementations (choice of evolution variable and kernel, momentum mappings, phase-space boundaries, massive quarks, photon emissions, etc.)
- All are tuned to data, and describe it reasonably well (typically better than expected from their formal accuracy)
- Some are (formally) more correct than others
  - However, not easy to assess accuracy for a general observable
  - Assessment (and improvement!) of formal accuracy is an active field of research