NLO QCD and EW corrections
(What are? How to compute them with MGS?)

Istituto Nazionale di Fisica Nucleare
SEZIONE DI BOLOGNA

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OUTLINE

Why NLO and higher-order corrections are important?

General aspects of NLO, focusing on NLO QCD

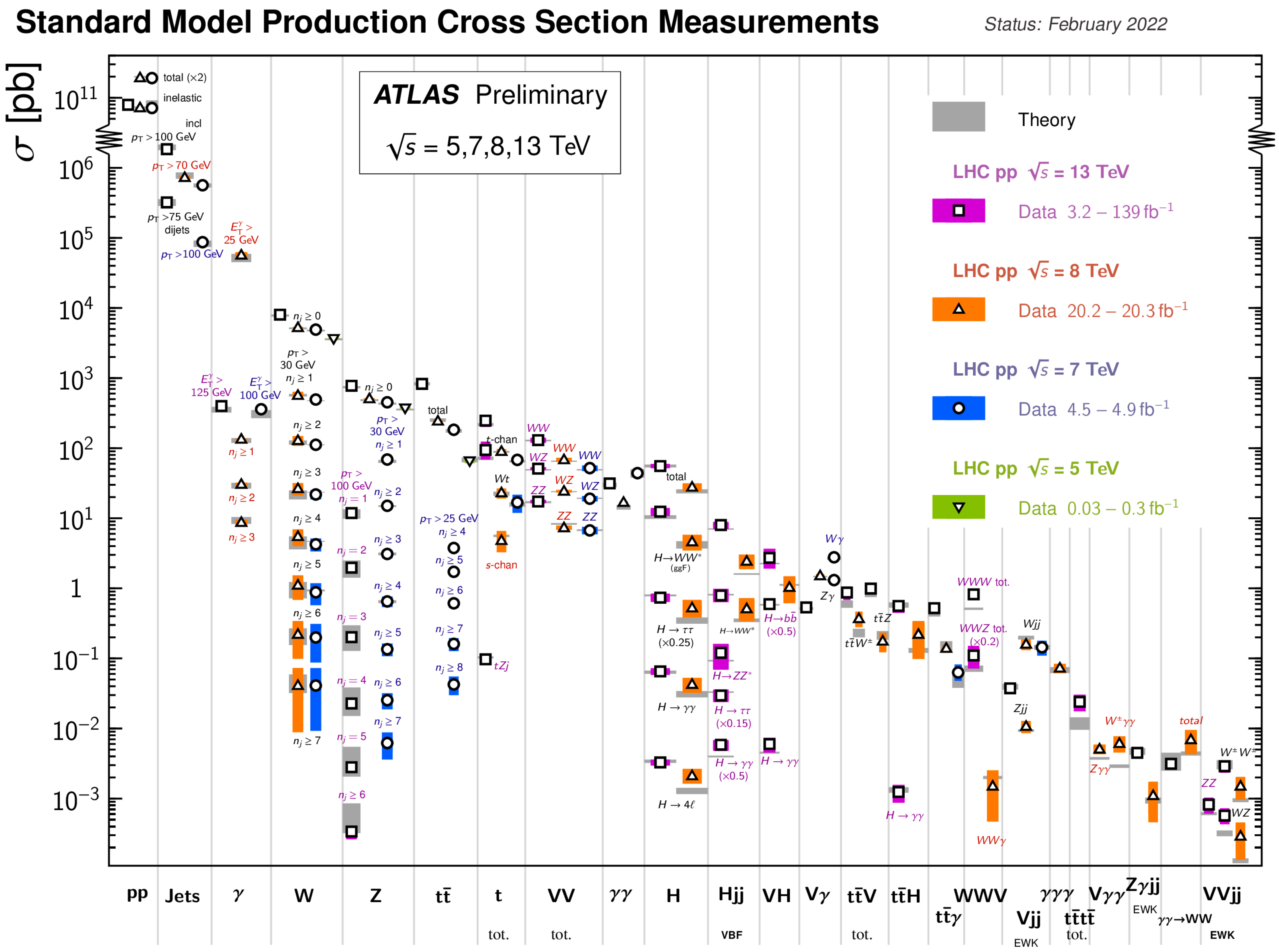
NLO EW (EW renormalisation)

If time is enough: Complete-NLO, Sudakov

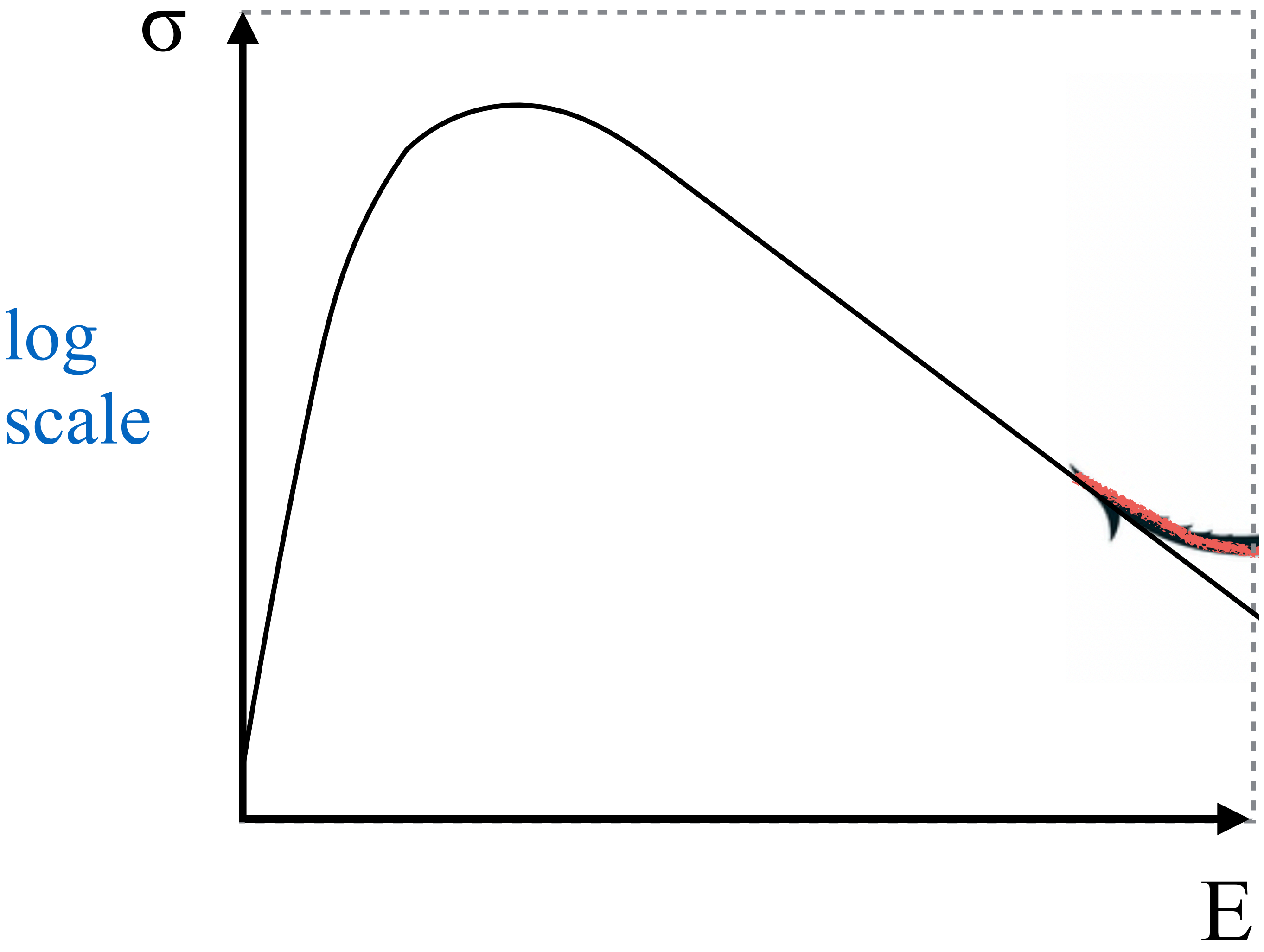
Motivations:

Why calculating NLO and higher-order corrections?

SM at the LHC (all good, too good)



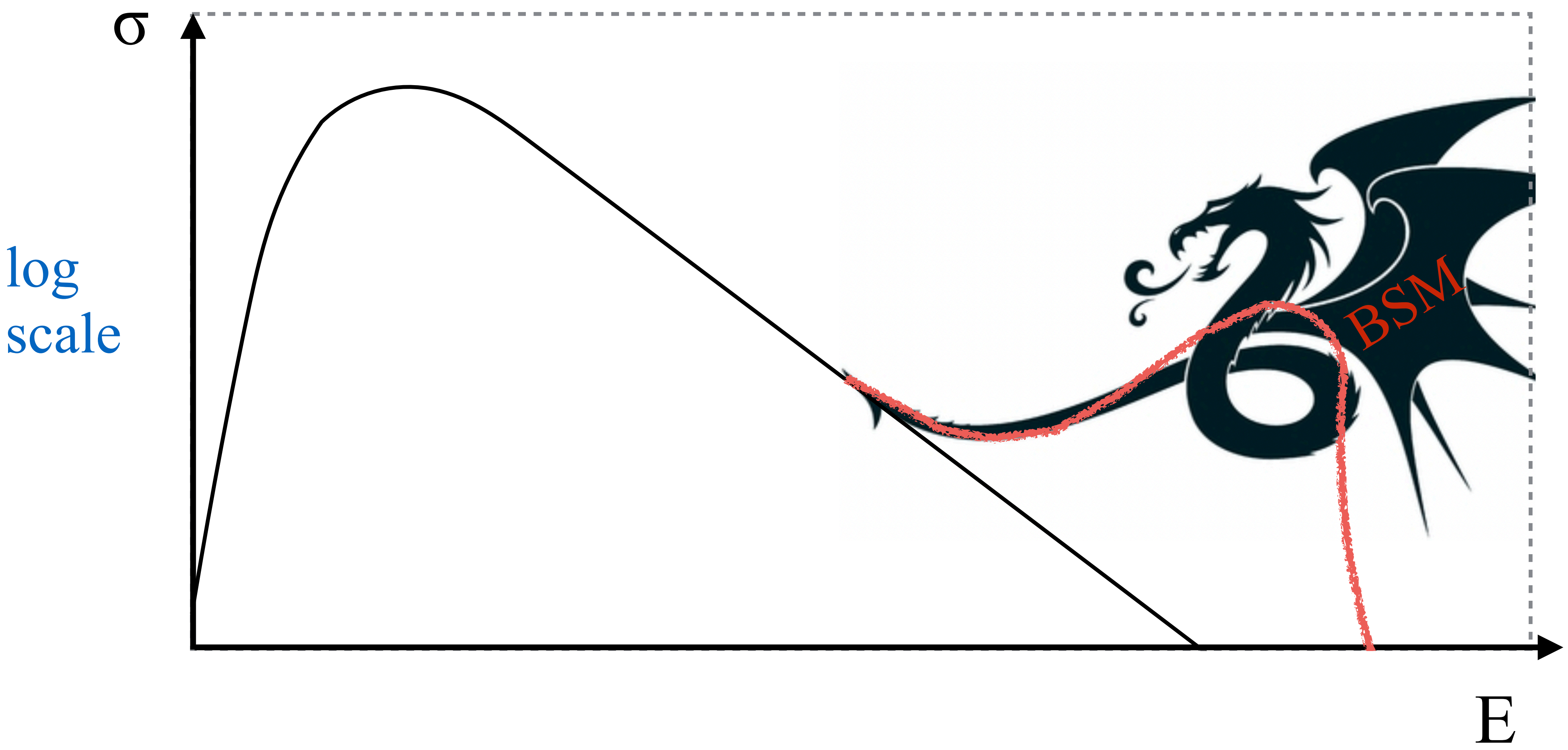
Differential distributions: let’s look at the tails



Small deviations from BSM dynamics?

With higher luminosity (and higher energy), at the LHC the accuracy of all measurements will in general increase, especially in the tail of distributions.

Differential distributions: let's look at the tails



With higher luminosity (and higher energy), at the LHC the accuracy of all measurements will in general increase, especially in the tail of distributions.

Precise predictions are necessary for the current and future measurements at the LHC. In order to match the experimental precision NLO QCD, NLO EW and even higher-order corrections are paramount.

Predictions at the LHC

Every prediction at the LHC starts form here:

Renormalization/factorization scale

$$\sigma_{H_1,H_2}(p_1,p_2) = \sum_{i,j} \int dx_1\, dx_2\, \boxed{f_i^{(H_1)}(x_1,\mu)\; f_j^{(H_2)}(x_2,\mu)}\, \boxed{\hat{\sigma}_{ij}(x_1 p_1, x_2 p_2, \alpha_S(\mu), \mu)}$$

PDFs

Partonic cross sections

- PDFs are fitted from experimental measurements, only the dependence on μ can be calculated in perturbation theory via DGLAP equations.
- Partonic cross sections can be calculated in perturbation theory via Feynman diagrams.

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- PDFs are fitted from experimental measurements, only the dependence on μ can be calculated in perturbation theory via DGLAP equations.
- Partonic cross sections can be calculated in perturbation theory via Feynman diagrams.

Precise predictions at the LHC: for what?

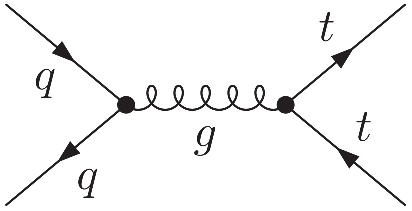
- More precise predictions for the total cross sections. (Total normalization)
- More precise differential distributions. (Kinematic-dependent corrections)
- Reduction of μ dependence. (Theoretical accuracy)

Methods/ Approximations

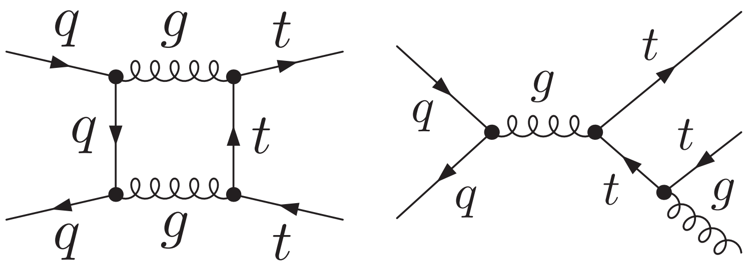
Fixed orders, Resummation, RGE, Parton Shower, Matching, Merging ………….

Fixed Order calculations

In the SM, contributions to the partonic cross section can be organized according to the powers of α_s and α (number of loop corrections and real emissions).

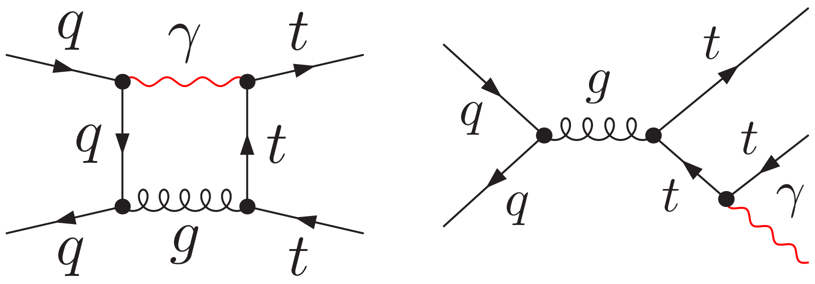


Born LO



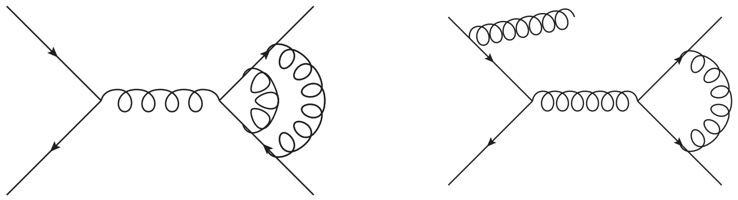
NLO QCD

$\mathcal{O}(\alpha_s)$ corrections



NLO EW

$\mathcal{O}(\alpha)$ corrections



NNLO QCD

$\mathcal{O}(\alpha_s^2)$ corrections

NNLO EW,

NNNLO QCD

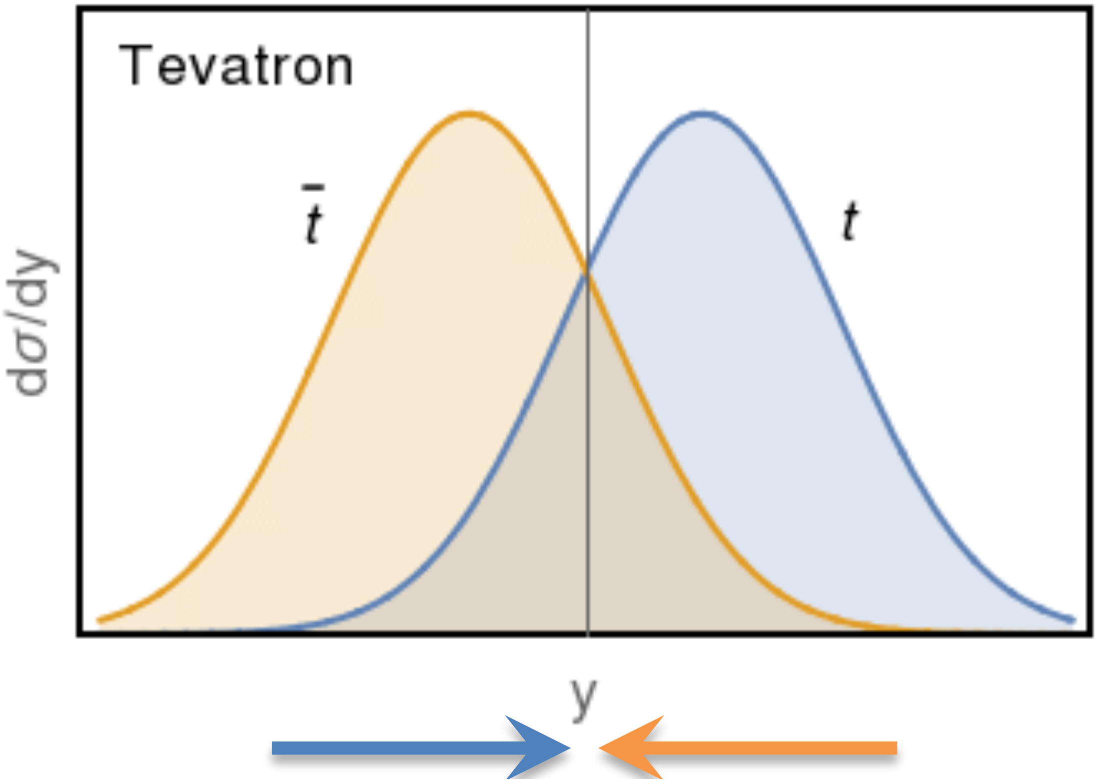
…..

At the LHC, QCD is everywhere.
Nowadays, a "standard" prediction in the SM is at NLO QCD accuracy.

NNLO QCD is expected to be of the same order of NLO EW $\alpha_s^2 \sim \alpha$.

Correct interpretation of the (B)SM signal

A recent story from an other hadron collider: **the top-quark forward-backward asymmetry at the Tevatron.**



proton antiproton

$$A_{FB}^{p\bar{p}} = \frac{\sigma(y_t > 0) - \sigma(y_t < 0)}{\sigma(y_t > 0) + \sigma(y_t < 0)}$$

D0 and especially CDF measured values for the forward-backward asymmetry that are larger than the SM prediction.

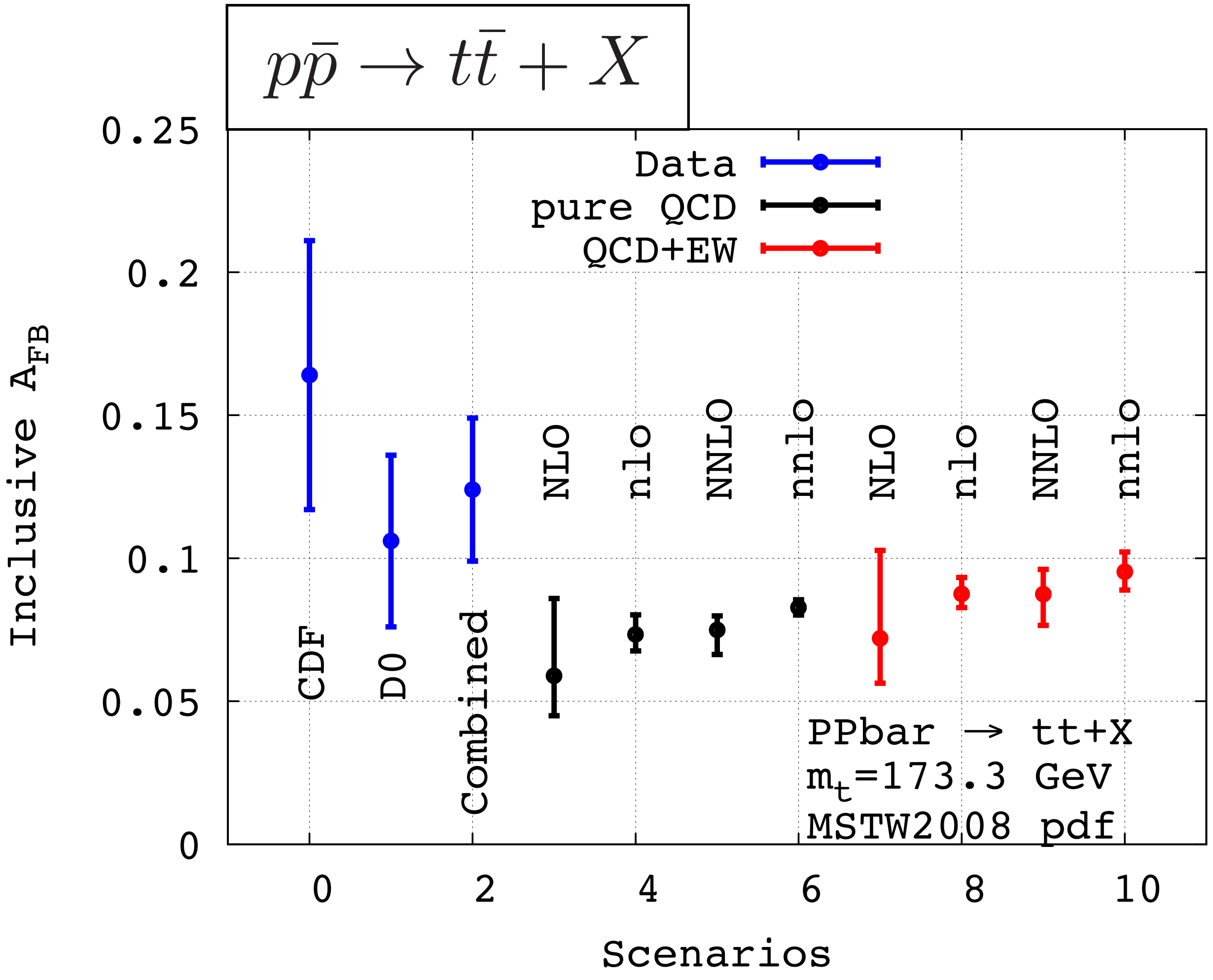
But which SM prediction?

Correct interpretation of the (B)SM signal

A recent story from an other hadron collider: **the top-quark forward-backward asymmetry at the Tevatron.**

Surprisingly (No Sudakov enhancement), the NLO EW induces corrections of order 20-25%.

DP, Hollik '11

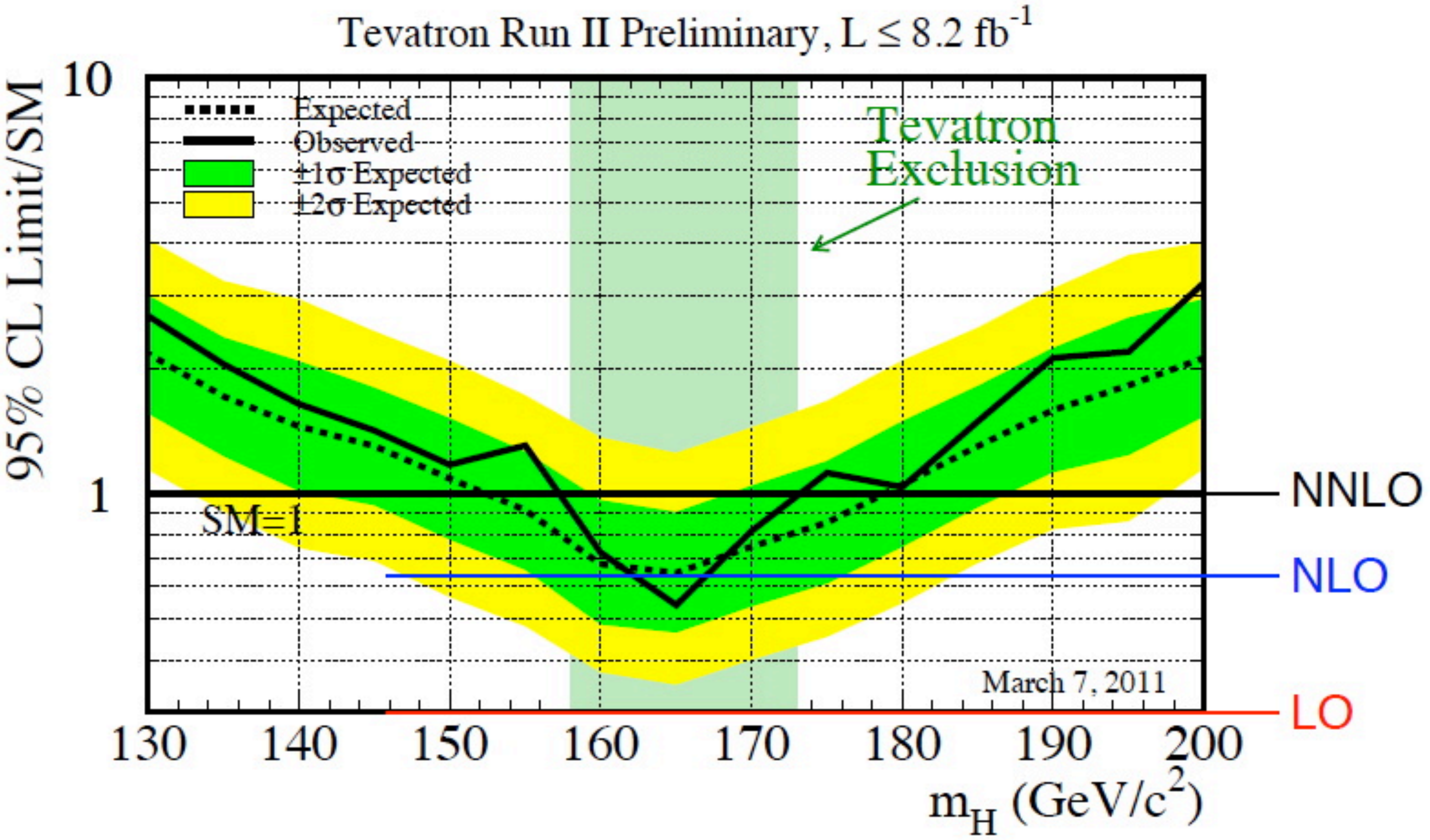


NNLO QCD and NLO EW are essential for a reliable theoretical prediction.

Missing higher-orders in the theoretical predictions may be misinterpreted as BSM signals.

Czakon, Fiedler, Mitov '14

Importance of NLO and NNLO QCD corrections



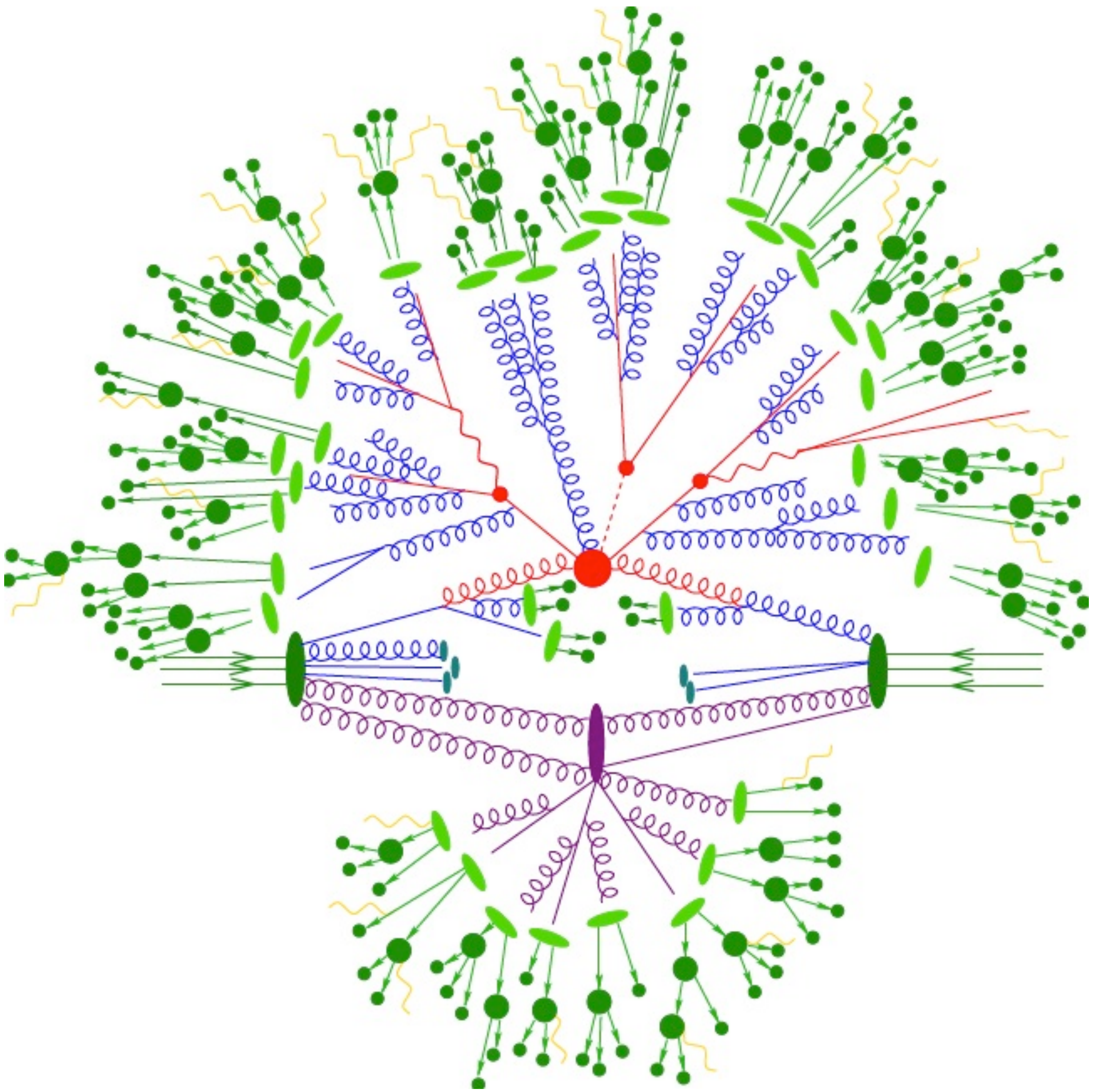
be careful : just illustrative example, not very precise

NLO Corrections

How do I calculate them?

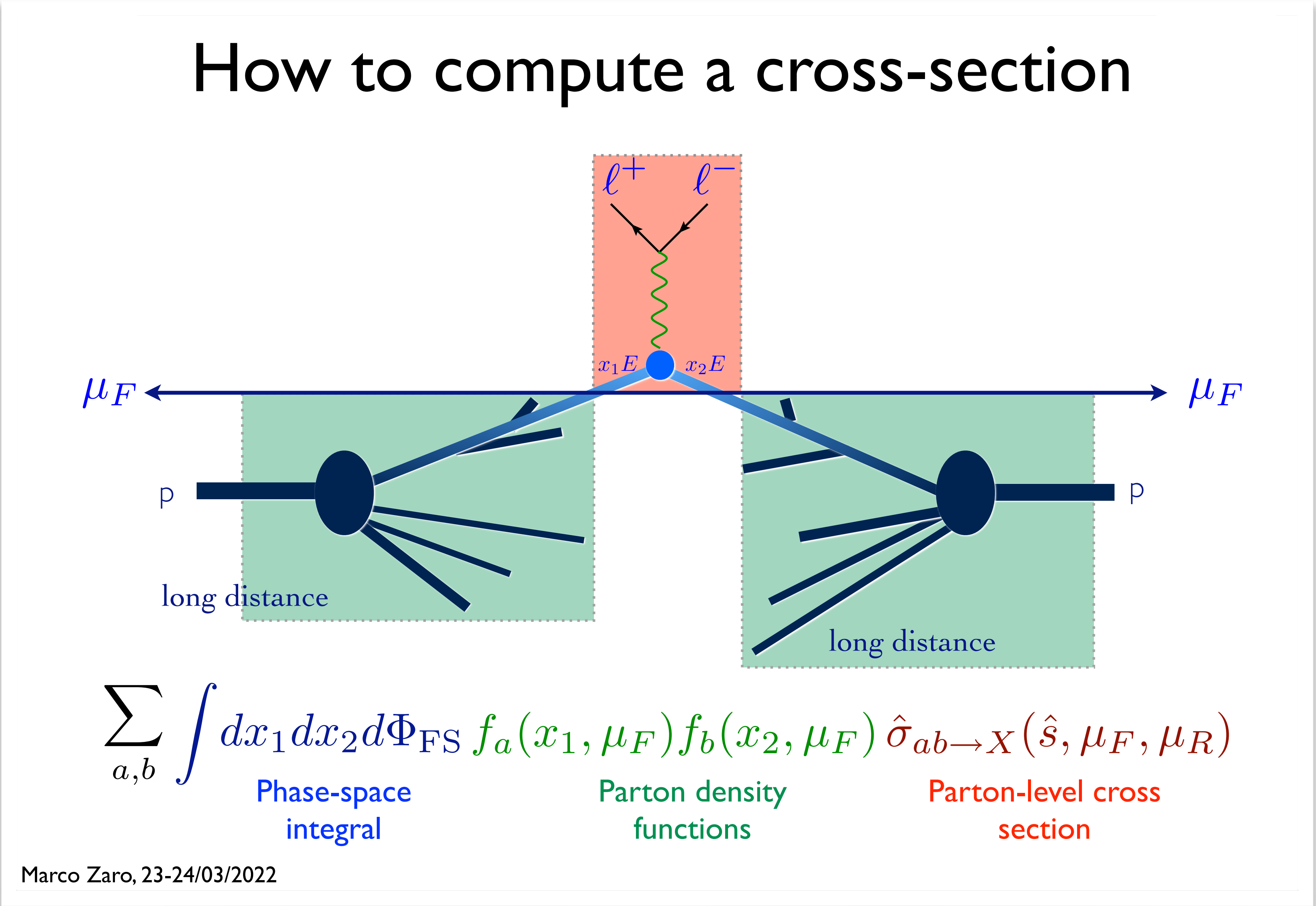
you can do it with MadGraph ….
but you need to know what’s going on in
order to understand the results

This is how an event at the LHC looks like:



We are going to discuss how to calculate with higher precision the central part of this picture: THE HARD SCATTERING PROCESS

The hard scattering process



How to compute a cross-section

$$\sum_{a,b}\int dx_1 dx_2 d\Phi_{\mathrm{FS}}\, f_a(x_1,\mu_F) f_b(x_2,\mu_F)\, \hat{\sigma}_{ab\to X}(\hat{s},\mu_F,\mu_R)$$

Phase-space integral

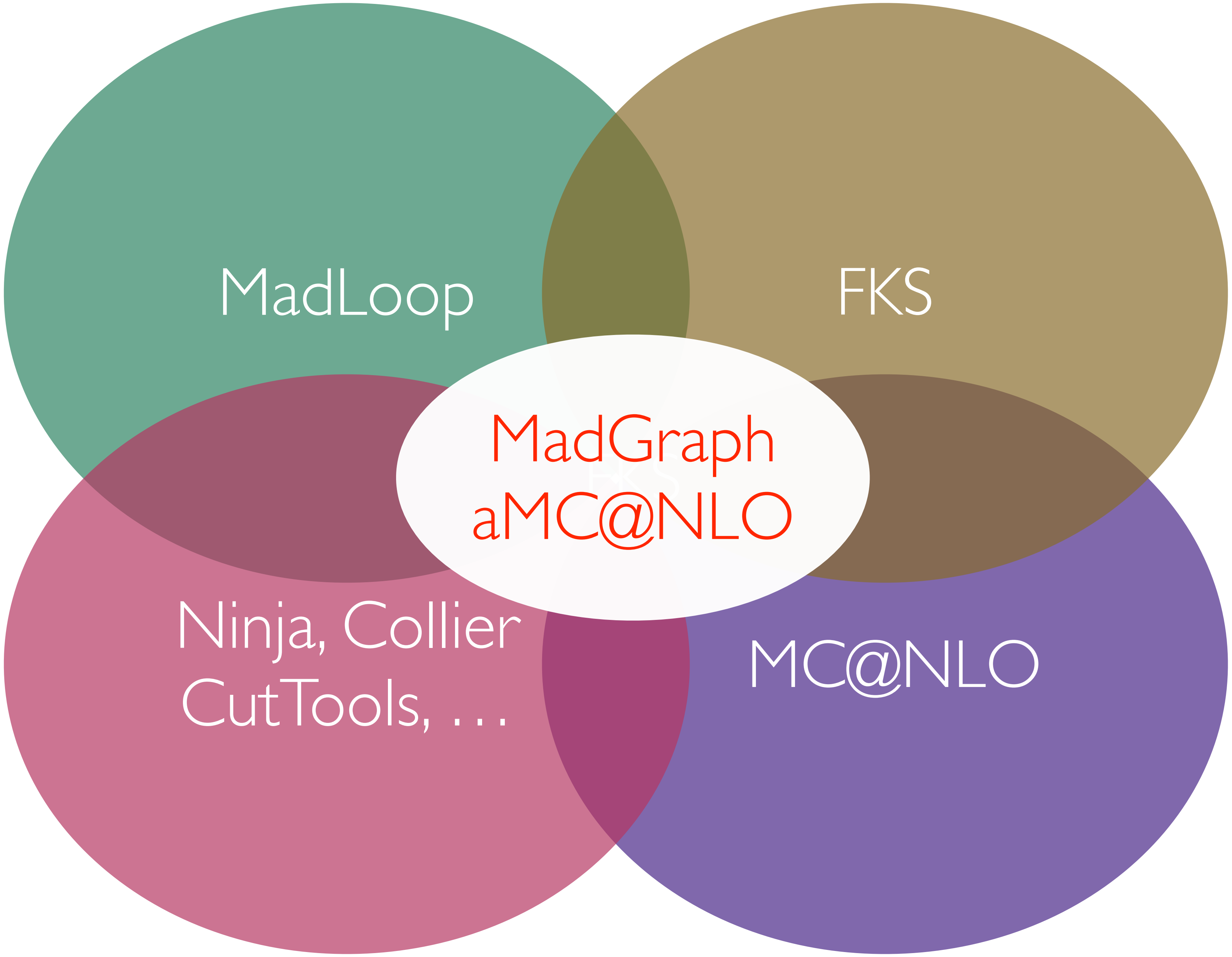
Parton density functions

Parton-level cross section

Marco Zaro, 23-24/03/2022

Actually we will mostly focus on how to improve the evaluation of the parton-level cross section, but also PDFs are equivalently important.

The structure of MadGraph resembles the steps of a calculation that in principle one could do with pencil and paper:



Alwall, Frederix, Frixione, Hirschi, Maltoni, Mattelaer, Shao, Stelzer, Torrielli, Zaro ’14
Frederix, Frixione, Hirschi, DP, Shao, Zaro ‘18

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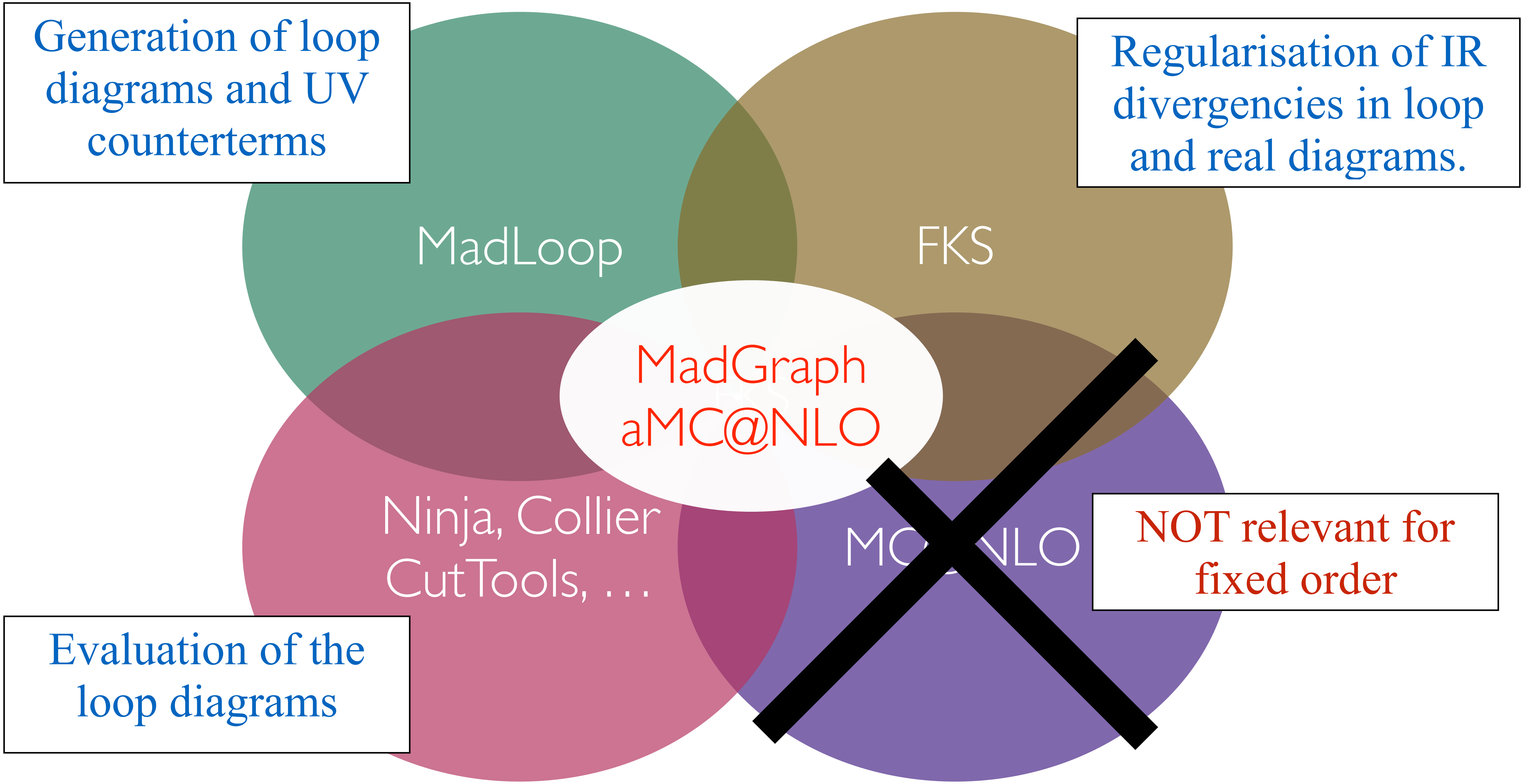


Generation of loop diagrams and UV counterterms

Regularisation of IR divergencies in loop and real diagrams.

MadLoop

FKS

MadGraph
aMC@NLO

Ninja, Collier
CutTools, …

MC@NLO

NOT relevant for fixed order

Evaluation of the loop diagrams

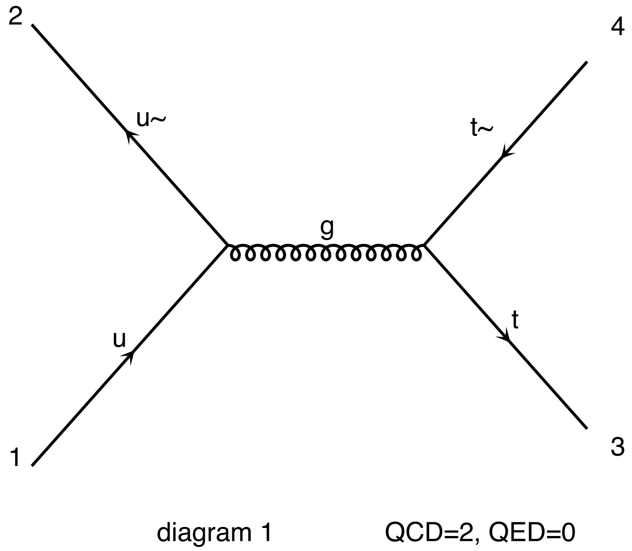
Alwall, Frederix, Frixione, Hirschi, Maltoni, Mattelaer, Shao, Stelzer, Torrielli, Zaro ’14
Frederix, Frixione, Hirschi, DP, Shao, Zaro ‘18

QCD corrections

(let’s forget about EW until further notice)

Let's start with a concrete example: $pp \to t\bar{t}$

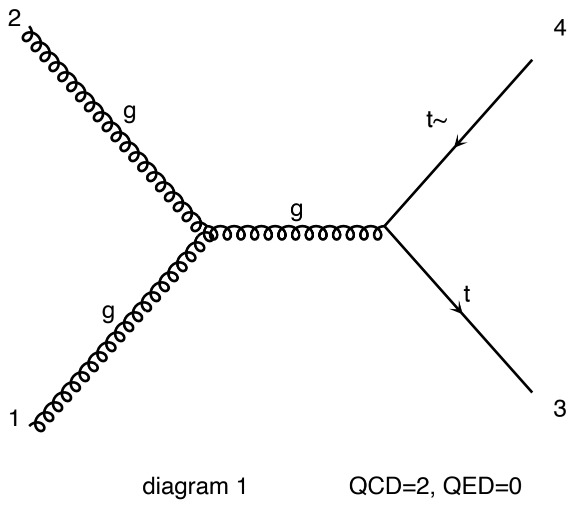
The LO cross section originates from the simplest diagrams you can imagine for all the possible partonic processes stemming from the partons in the proton: $q\bar{q} \to t\bar{t}$ and $gg \to t\bar{t}$.

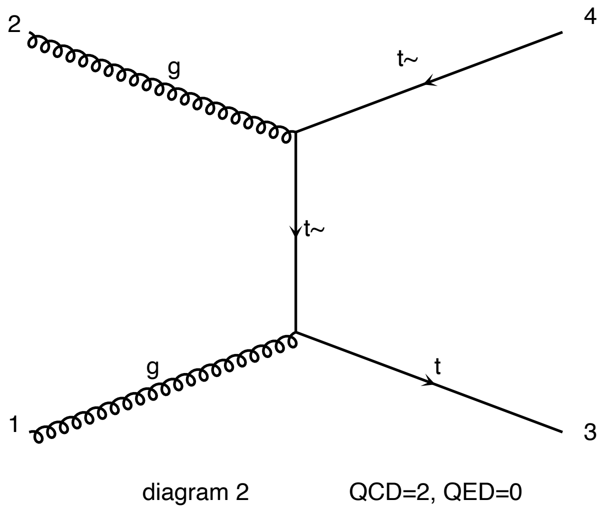


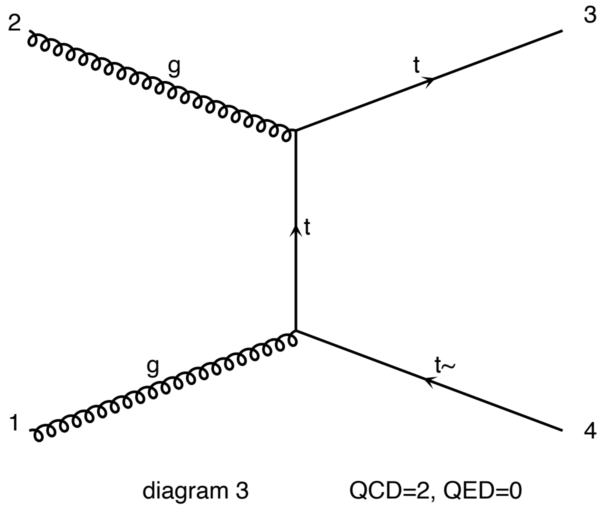
These are the diagrams you get via the commands in MG5:

```
generate p p > t t~
output ttbarLO_folder
```

and you can then calculate the cross section:







$pp \to t\bar{t}$ at LO

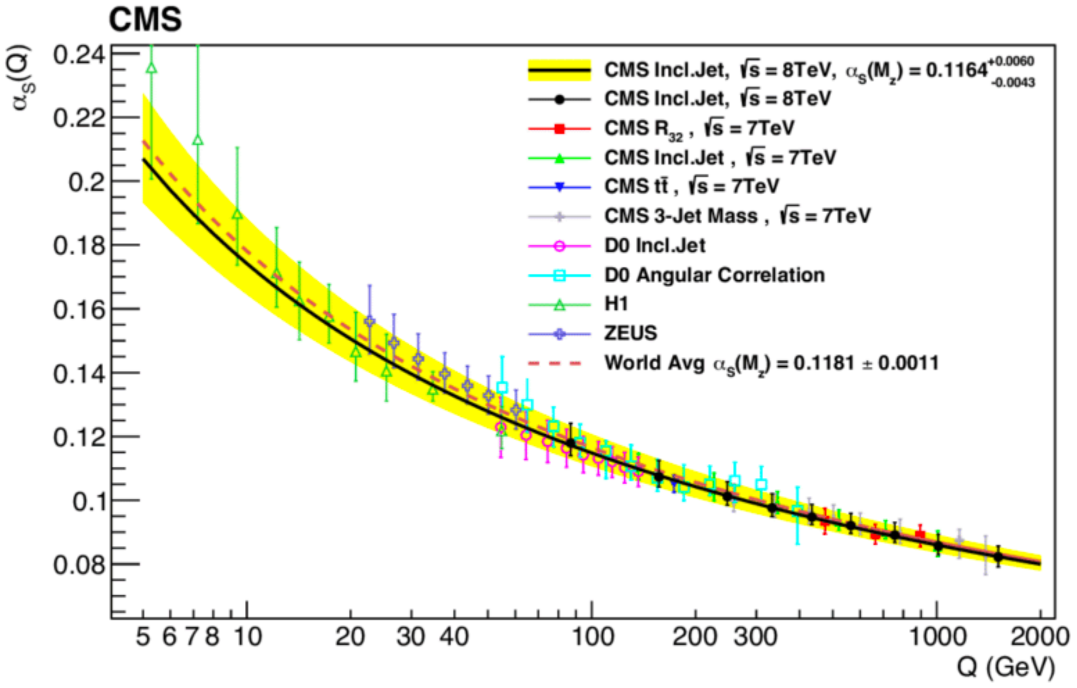
Easy calculation that can be done also with pencil and paper, for example for the $q\bar{q}$ initial state (gg case is similar):

$$\frac{d\sigma}{d\Omega}(q\bar{q} \to Q\bar{Q}) = \frac{\boxed{\alpha_s^2}}{9s^3}\sqrt{1 - \frac{4m_Q^2}{s}}$$

$$\left[(m_Q^2 - t)^2 + (m_Q^2 - u)^2 + 2m_Q^2 s\right],$$

Notice that the cross sections is proportional to $\boxed{\alpha_s^2}$, as the squared of the amplitudes I have shown before.

Remember that α_s is not a constant and it runs, so your cross section will depend on which scale you will choose for it.



$pp \to t\bar{t}$ at NLO QCD

For a given final state if LO $\sim \alpha_s^n$, NLO QCD corrections are all the contributions proportional to the ***inclusive*** production of the same final state at the order α_s^{n+1}.

In the case of $pp \to t\bar{t}$ this means: $pp \to t\bar{t}(+X)$ of order α_s^3.

In MG5 this is equivalent to ask

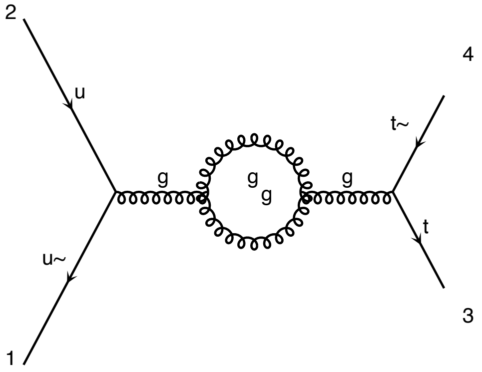
```
generate p p > t t~ [QCD]
output ttbarNLO_folder
```

and you can then calculate the cross section:

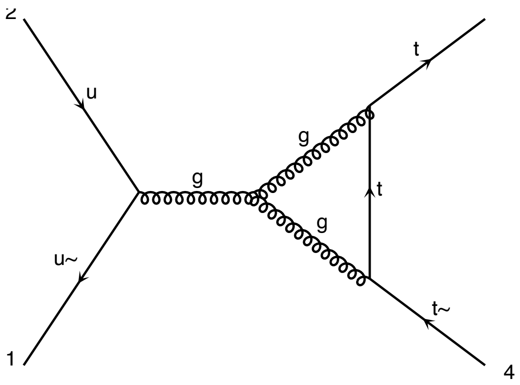
But let's see some diagrams that emerge at this order.

$pp \rightarrow t\bar{t}$ at NLO QCD

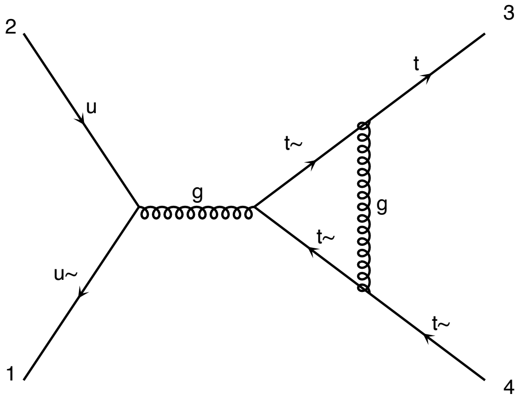
one-loop diagrams….



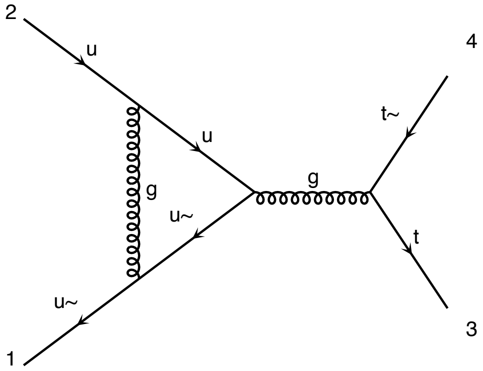
loop diagram 1 QCD=4, QED=0



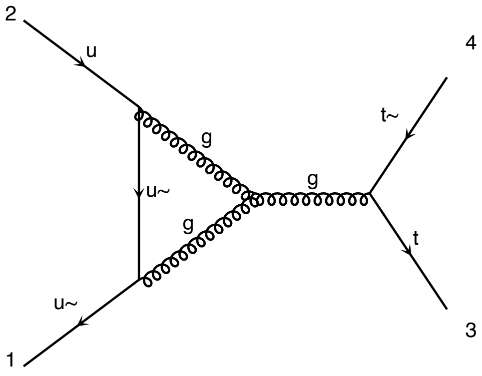
loop diagram 2 QCD=4, QED=0



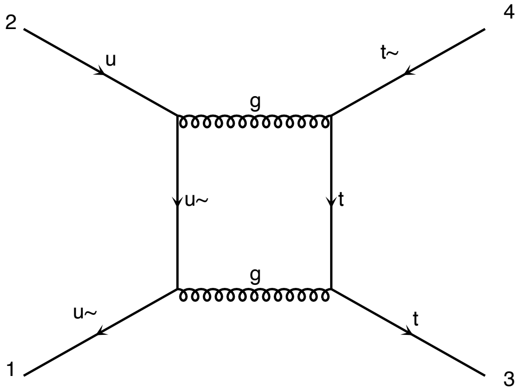
loop diagram 3 QCD=4, QED=0



loop diagram 4 QCD=4, QED=0

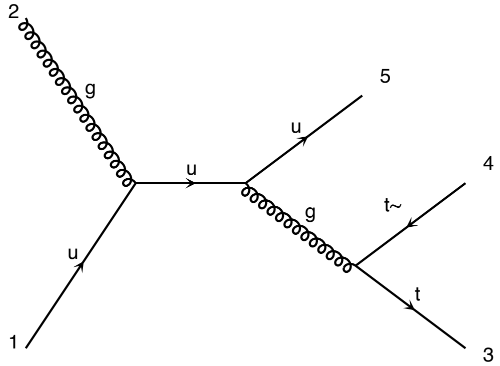


loop diagram 5 QCD=4, QED=0

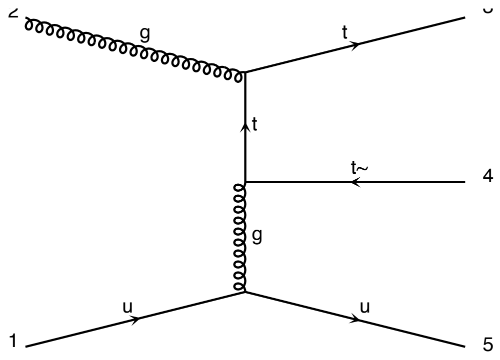


loop diagram 6 QCD=4, QED=0

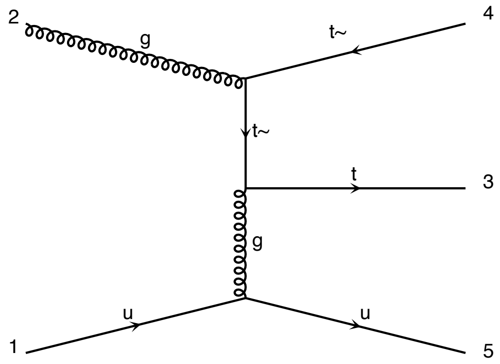
but also real-emission ones



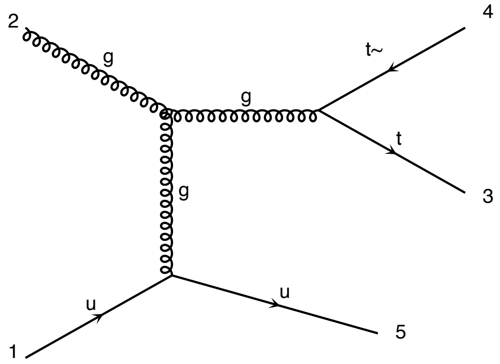
real diagram 1 QCD=3, QED=0



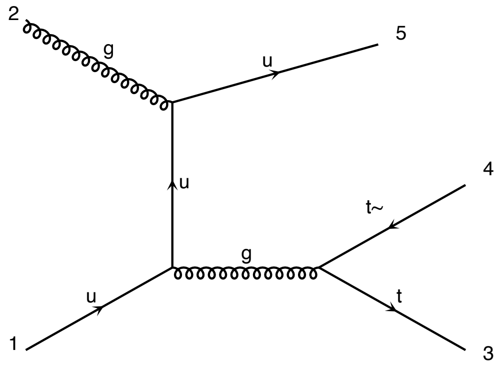
real diagram 2 QCD=3, QED=0



real diagram 3 QCD=3, QED=0



real diagram 4 QCD=3, QED=0



real diagram 5 QCD=3, QED=0

Notice that I have shown a case where the initial state does not even show up at LO.

$pp \to t\bar{t}$ at NLO QCD (anatomy of the diagrams)

LO $\mathcal{O}(\alpha_s^2)$

$$q\bar{q} \to t\bar{t} : \; |\mathscr{M}_{\text{tree}}|^2$$

$$gg \to t\bar{t} : \; |\mathscr{M}_{\text{tree}}|^2$$

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Expansion at order $\mathcal{O}(\alpha_s^2)$ or $\mathcal{O}(\alpha_s^3)$ of

$$|\mathscr{M}|^2 = \left|\sum_{i=0}^{\infty} \mathscr{M}_{\text{i-loop}}\right|^2$$

NLO $\mathcal{O}(\alpha_s^3)$

$$\left.\begin{array}{l} q\bar{q} \rightarrow t\bar{t} : \; 2\Re(\mathscr{M}_{\text{tree}}\mathscr{M}^*_{\text{1-loop}}) \\ gg \rightarrow t\bar{t} : \; 2\Re(\mathscr{M}_{\text{tree}}\mathscr{M}^*_{\text{1-loop}}) \end{array}\right\} \text{Virtual}$$

$$\left.\begin{array}{l} q\bar{q} \rightarrow t\bar{t}g : \; |\mathscr{M}_{\text{tree}}|^2 \\ gg \rightarrow t\bar{t}g : \; |\mathscr{M}_{\text{tree}}|^2 \\ qg \rightarrow t\bar{t}q : \; |\mathscr{M}_{\text{tree}}|^2 \\ \bar{q}g \rightarrow t\bar{t}\bar{q} : \; |\mathscr{M}_{\text{tree}}|^2 \end{array}\right\} \text{Real emission}$$

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Expansion at order $\mathcal{O}(\alpha_s^2)$ or $\mathcal{O}(\alpha_s^3)$ of

$$|\mathcal{M}|^2 = \left|\sum_{i=0}^{\infty} \mathcal{M}_{\text{i−loop}}\right|^2$$

UV and IR divergent!

NLO $\mathcal{O}(\alpha_s^3)$

UV and IR finite

$$\left.\begin{array}{l} q\bar{q} \to t\bar{t} : \ 2\Re(\mathcal{M}_{\text{tree}}\mathcal{M}^*_{\text{1−loop}}) \\ gg \to t\bar{t} : \ 2\Re(\mathcal{M}_{\text{tree}}\mathcal{M}^*_{\text{1−loop}}) \end{array}\right\} \text{Virtual}$$

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IR divergent!

UV (ultraviolet) divergencies are canceled via renormalisation.

IR (infrared) divergencies are canceled via combination of virtual+real (KLN) and subtraction to the PDFs.

$pp \to t\bar{t}$ at NLO QCD (anatomy of the diagrams)

LO $\mathcal{O}(\alpha_s^2)$

$q\bar{q} \to t\bar{t}: \ |\mathcal{M}_{\text{tree}}|^2$

$gg \to t\bar{t}: \ |\mathcal{M}_{\text{tree}}|^2$

Expansion at order $\mathcal{O}(\alpha_s^2)$ or $\mathcal{O}(\alpha_s^3)$ of

$$|\mathcal{M}|^2 = \left|\sum_{i=0}^{\infty} \mathcal{M}_{\text{i-loop}}\right|^2$$

UV and IR divergent!

NLO $\mathcal{O}(\alpha_s^3)$

UV and IR finite

$q\bar{q} \to t\bar{t}: \ 2\Re(\mathcal{M}_{\text{tree}}\mathcal{M}^*_{\text{1-loop}})$

$gg \to t\bar{t}: \ 2\Re(\mathcal{M}_{\text{tree}}\mathcal{M}^*_{\text{1-loop}})$

} Virtual

$q\bar{q} \to t\bar{t}g: \ |\mathcal{M}_{\text{tree}}|^2$

$gg \to t\bar{t}g: \ |\mathcal{M}_{\text{tree}}|^2$

$qg \to t\bar{t}q: \ |\mathcal{M}_{\text{tree}}|^2$

$\bar{q}g \to t\bar{t}\bar{q}: \ |\mathcal{M}_{\text{tree}}|^2$

} Real emission

IR divergent!

UV (ultraviolet) divergencies are canceled via renormalisation.

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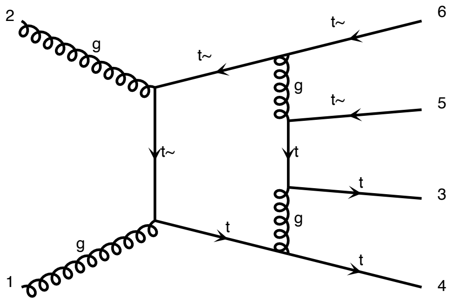
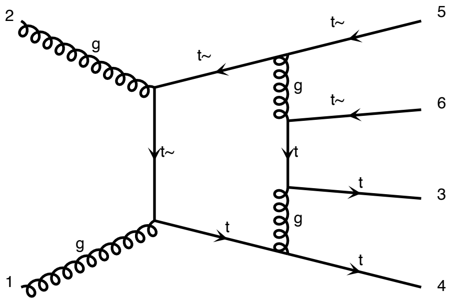
NOTE THAT WITHIN $t\bar{t}$ AT NLO WE HAVE ALSO $t\bar{t}$ + 1 jet AT LO!

Loop Diagrams

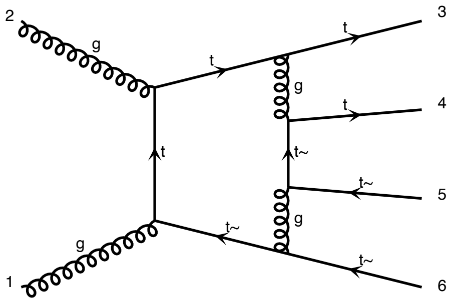
Before looking at renormalisation, let's see how loop diagrams are computed in MadGraph.
While for a process like $pp \to t\bar{t}$ one may even use textbook methods, this approach would not in general work for complex processes and especially for the automation.



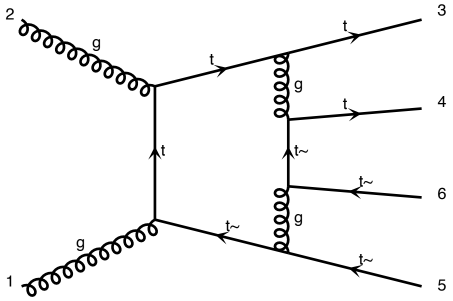
loop diagram 1639 QCD=6, QED=0



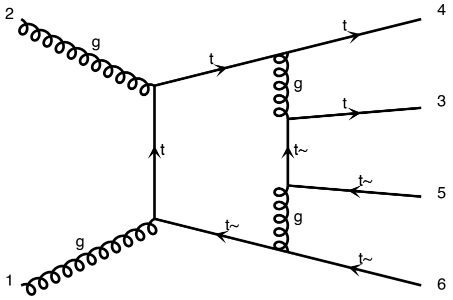
loop diagram 1640 QCD=6, QED=0



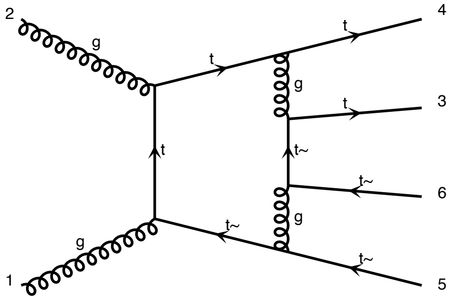
loop diagram 1641 QCD=6, QED=0



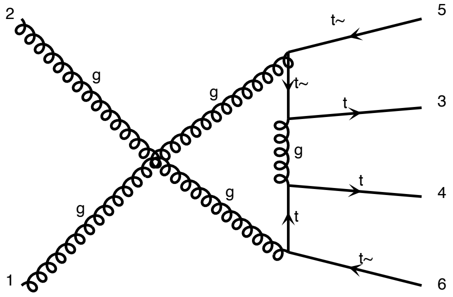
loop diagram 1642 QCD=6, QED=0



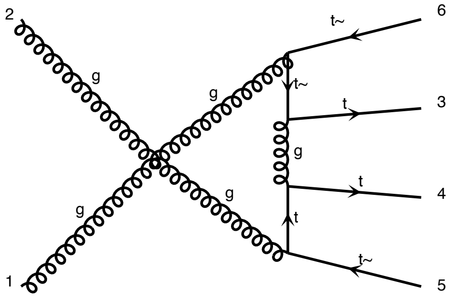
loop diagram 1643 QCD=6, QED=0



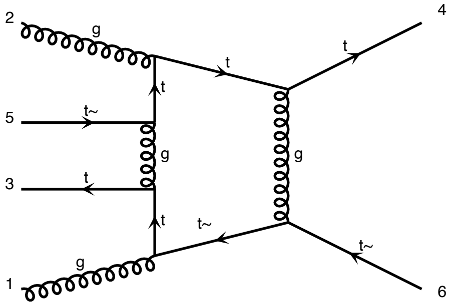
loop diagram 1644 QCD=6, QED=0



loop diagram 1645 QCD=6, QED=0



loop diagram 1646 QCD=6, QED=0



loop diagram 1647 QCD=6, QED=0

These are just 9 of the 2478 diagrams entering the 1-loop amplitude for $gg \to t\bar{t}t\bar{t}$.

Try just to calculate one by hand (notice that there are 6 propagators!) and you can realise the challenge ..

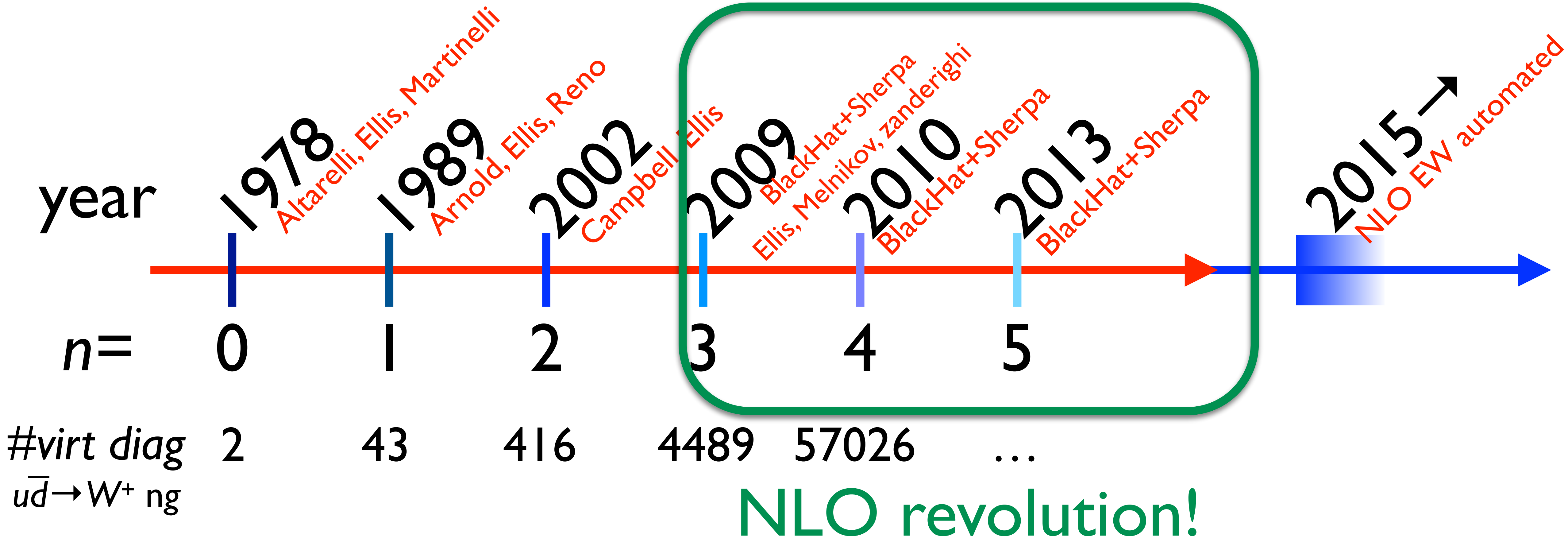
Next slides are stolen from Marco Zaro, who gave this lecture at the 2022 edition of this school.

Loops

how to calculate them

NLO (pre)history

- NLO evolution:
  - e.g. pp→W+*n* jets



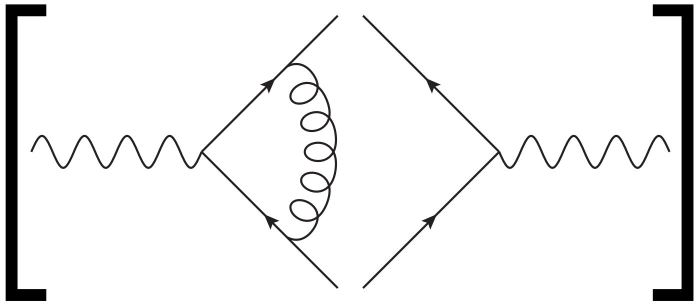
year	1978	1989	2002	2009	2010	2013	2015→
	Altarelli, Ellis, Martinelli	Arnold, Ellis, Reno	Campbell, Ellis	BlackHat+Sherpa Ellis, Melnikov, zanderighi	BlackHat+Sherpa	BlackHat+Sherpa	NLO EW automated
n=	0	1	2	3	4	5	
#virt diag $u\bar{d}\to W^+$ ng	2	43	416	4489	57026	…	

NLO revolution!

Computing loops numerically

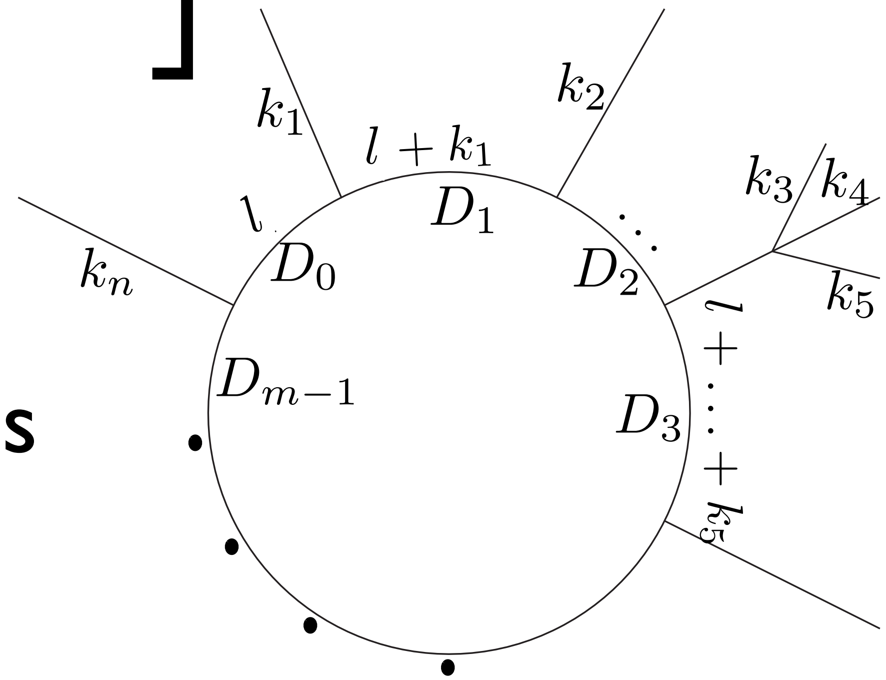
- Consider a m-point one-loop diagram with n external momenta

$$d\sigma_V = 2\Re\Bigg[$$



$$\Bigg]$$

- The integral to compute is



$$p_1 = k_1$$

$$p_2 = k_2$$

$$p_3 = k_3 + k_4 + k_5$$

$$\int d^d l \frac{N(l)}{D_0 D_1 \ldots D_{m-1}}$$

$$D_i = (l + p_i)^2 - m_i^2$$

A hint…

- Any one-loop integral can be cast in the form

$$\int d^d l \frac{N(l)}{D_0 D_1 \dots D_{m-1}} = \sum \mathrm{coeff}_i \int d^d l \frac{1}{D_{i_0} D_{i_1} \dots}$$

- It is a linear combination of scalar integrals
- If d=4+ε, only scalar integrals with up to 4 denominators are needed → the basis is finite!
- The coefficients depend only on external momenta and parameters

Scalar integrals

- Scalar integrals are known and available as libraries

FF (van Oldenborgh, CPC 66,1991)
QCDLoop (Ellis, Zanderighi, arXiv:0712.1851)
OneLOop (Van Hameren, arXiv:1007.4716)

$$\mathcal{M}^{1\text{loop}} = \sum_{i_0,i_1,i_2,i_3} d_{i_0i_1i_2i_3}\mathcal{D}_{i_0i_1i_2i_3} + \sum_{i_0,i_1,i_2} c_{i_0i_1i_2}\mathcal{C}_{i_0i_1i_2} + \sum_{i_0,i_1} b_{i_0i_1}\mathcal{B}_{i_0i_1} + \sum_{i_0} a_{i_0}\mathcal{A}_{i_0}$$

Box $\mathcal{D}_{i_0i_1i_2i_3} = \int d^dl \frac{1}{D_{i_0}D_{i_1}D_{i_2}D_{i_3}}$

Triangle $\mathcal{C}_{i_0i_1i_2} = \int d^dl \frac{1}{D_{i_0}D_{i_1}D_{i_2}}$

Bubble $\mathcal{B}_{i_0i_1} = \int d^dl \frac{1}{D_{i_0}D_{i_1}}$

Tadpole $\mathcal{A}_{i_0} = \int d^dl \frac{1}{D_{i_0}}$

How to compute the coefficients?

- Several techniques exist
  - Computation of loop MEs
    - Tensor reduction
    - Generalized unitarity
    - Integrand reduction

Passarino, Veltman, 1979
Denner, Dittmaier, hep-ph/509141
Binoth, Guillet, Heinrich, Pilon, Reiter, arXiv:0810.0992

Bern, Dixon, Dunbar, Kosower, hep-ph/9403226 + …
Ellis, Giele, Kunszt, arXiv:0708.2398
+ Melnikov, arXiv:0806.3467

Ossola, Papadopoulos, Pittau, hep-ph/0609007
Del Aguila, Pittau, hep-ph/0404120
Mastrolia, Ossola, Reiter, Tramontano, arXiv:1006.0710

Integrand reduction

- Can we take away the integral?

$$\int d^d l \frac{N(l)}{D_0 D_1 \dots D_{m-1}} = \sum \text{coeff}_i \int d^d l \frac{1}{D_{i_0} D_{i_1} \dots}$$

- Of course not, we must take into account for terms which integrate to 0, the so-called spurious terms:

$$\frac{N(l)}{D_0 D_1 \dots D_{m-1}} \neq \sum \text{coeff}_i \frac{1}{D_{i_0} D_{i_1} \dots}$$

$$\frac{N(l)}{D_0 D_1 \dots D_{m-1}} = \sum \left(\text{coeff}_i + \text{spurious}_i(l)\right) \frac{1}{D_{i_0} D_{i_1} \dots}$$

Spurious terms

- The functional form of the spurious terms is known and depends on the rank (powers of l in the numerator) and on the number of denominators Del Aguila, Pittau, hep-ph/0404120
  - E.g. a rank-1 box

$$\tilde{d}_{i_0 i_1 i_2 i_3}(l) = \tilde{d}_{i_0 i_1 i_2 i_3}\, \epsilon^{\mu\nu\rho\sigma}\, l^{\mu} p_1^{\nu} p_2^{\rho} p_3^{\sigma}$$

  - The integral is 0

$$\int d^d l \frac{\tilde{d}_{i_0 i_1 i_2 i_3}(l)}{D_0 D_1 D_2 D_3} = \tilde{d}_{i_0 i_1 i_2 i_3} \int d^d l \frac{\epsilon^{\mu\nu\rho\sigma}\, l^{\mu} p_1^{\nu} p_2^{\rho} p_3^{\sigma}}{D_0 D_1 D_2 D_3} = 0$$

OPP decomposition

Ossola, Papadopoulos, Pittau, hep-ph/0609007

$$\frac{N(l)}{D_0 D_1 \ldots D_{m-1}} = \sum \left(\text{coeff}_i + \text{spurious}_i(l)\right) \frac{1}{D_{i_0} D_{i_1} \ldots}$$

- If we multiply both sides times $D_0 D_1 \ldots D_{m\text{-}1}$ we get

$$\begin{aligned} N(l) = & \sum_{i_0,i_1,i_2,i_3} \left(d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}\right) \prod_{i \neq i_0,i_1,i_2,i_3} D_i \\ & + \sum_{i_0,i_1,i_2} \left(c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}\right) \prod_{i \neq i_0,i_1,i_2} D_i \\ & + \sum_{i_0,i_1} \left(b_{i_0 i_1} + \tilde{b}_{i_0 i_1}\right) \prod_{i \neq i_0,i_1} D_i \\ & + \sum_{i_0} \left(a_{i_0} + \tilde{a}_{i_0}\right) \prod_{i \neq i_0} D_i \\ & + \tilde{P}(l) \prod_i D_i + \mathcal{O}(\varepsilon) \end{aligned}$$

Getting the coefficients

- *N(l)* is known from the diagrams and the functional form of spurious terms is known too
  - We can sample *N(l)* at various values of the loop momentum, and get a system of linear equations
  - The sampling can be done numerically
  - By choosing smart values of *l* (in the complex plane), the system can be greatly simplified
  - E.g. we can choose l such that

$$D_1(l^{\pm}) = D_2(l^{\pm}) = D_3(l^{\pm}) = D_4(l^{\pm}) = 0$$

$$\Downarrow$$

$$N(l^{\pm}) = (d_{1234} + \tilde{d}_{1234}(l^{\pm})) \prod_{i \neq 1,2,3,4} D_i(l^{\pm})$$

Getting the coefficient: recap

- For each PS point, we have to solve a system of equations numerically
- The system reduces when special values of the loop momentum are chosen
- $N(l)$ can be the numerator of the full matrix element, of a single diagram or anything in between
- For a given PS point, the numerator has to be sampled several times (~50 for a 4-point diagrams)

The evil is in the *d*etails: Complications in *d* dimensions

- So far, we did not care much about the number of dimensions we were using
- In general, external momenta and polarisations are in 4 dimensions; only the loop momentum is in d
- To be more rigorous, we compute the integral

$$\int d^d l \frac{N(l,\tilde{l})}{\bar{D}_0 \bar{D}_1 \ldots \bar{D}_{m-1}}$$

$$\underset{d\text{-dim}}{\bar{l}} = \underset{4\text{-dim}}{l} + \underset{\varepsilon\text{-dim}}{\tilde{l}}$$

$$\bar{D}_i = (\bar{l} + p_i)^2 - m_i = (l + p_i)^2 - m_i^2 + \tilde{l}^2 = D_i + \tilde{l}^2$$

$$l \cdot \tilde{l} = 0 \qquad \bar{l} \cdot p_i = l \cdot p_i \qquad \bar{l} \cdot \bar{l} = l \cdot l + \tilde{l} \cdot \tilde{l}$$

Implications

- The reduction should be consistently done in d dimensions

$$
\begin{aligned}
\mathcal{M}^{\rm 1loop} = & \sum_{i_0,i_1,i_2,i_3} d_{i_0 i_1 i_2 i_3} \bar{\mathcal{D}}_{i_0 i_1 i_2 i_3} \\
& + \sum_{i_0,i_1,i_2} c_{i_0 i_1 i_2} \bar{\mathcal{C}}_{i_0 i_1 i_2} \\
& + \sum_{i_0,i_1} b_{i_0 i_1} \bar{\mathcal{B}}_{i_0 i_1} \\
& + \sum_{i_0} a_{i_0} \bar{\mathcal{A}}_{i_0} \\
& + \mathcal{O}(\varepsilon)
\end{aligned}
$$

Implications

- The reduction should be consistently done in d dimensions

$$\begin{aligned}\mathcal{M}^{1\text{loop}} = &\sum_{i_0,i_1,i_2,i_3} d_{i_0i_1i_2i_3}\bar{\mathcal{D}}_{i_0i_1i_2i_3}\\ &+\sum_{i_0,i_1,i_2} c_{i_0i_1i_2}\bar{\mathcal{C}}_{i_0i_1i_2}\\ &+\sum_{i_0,i_1} b_{i_0i_1}\bar{\mathcal{B}}_{i_0i_1}\\ &+\sum_{i_0} a_{i_0}\bar{\mathcal{A}}_{i_0}\\ &+\mathcal{O}(\varepsilon)\end{aligned}$$

$$\Longrightarrow$$

$$\begin{aligned}\mathcal{M}^{1\text{loop}} = &\sum_{i_0,i_1,i_2,i_3} d_{i_0i_1i_2i_3}\mathcal{D}_{i_0i_1i_2i_3}\\ &+\sum_{i_0,i_1,i_2} c_{i_0i_1i_2}\mathcal{C}_{i_0i_1i_2}\\ &+\sum_{i_0,i_1} b_{i_0i_1}\mathcal{B}_{i_0i_1}\\ &+\sum_{i_0} a_{i_0}\mathcal{A}_{i_0}\\ &+\enclose{circle}{R}+\mathcal{O}(\varepsilon)\end{aligned}$$

That is why the *rational terms* are needed

The rational terms

OPP, arXiv:0802.1876

- In the OPP method, two types of rational terms are there: R=R_1+R_2
- Both originate from the UV part of the model, but only R_1 can be computed in the OPP decomposition
- R_1 originates from the *denominators* (propagators) in the loops

$$\frac{1}{\bar{D}_i} = \frac{1}{D_i}\left(1 - \frac{\tilde{l}^2}{D_i}\right)$$

- The denominator structure is known, so these terms can be directly included in the OPP reduction
- R_1 contributions are proportional to

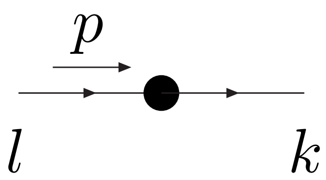
$$\int d^d l \frac{\tilde{l}^2}{\bar{D}_i \bar{D}_j} = -\frac{i\pi^2}{2}\left[m_i^2 + m_j^2 - \frac{(p_i - p_j)^2}{2}\right] + \mathcal{O}(\varepsilon)$$

$$\int d^d l \frac{\tilde{l}^2}{\bar{D}_i \bar{D}_j \bar{D}_k} = -\frac{i\pi^2}{2} + \mathcal{O}(\varepsilon)$$

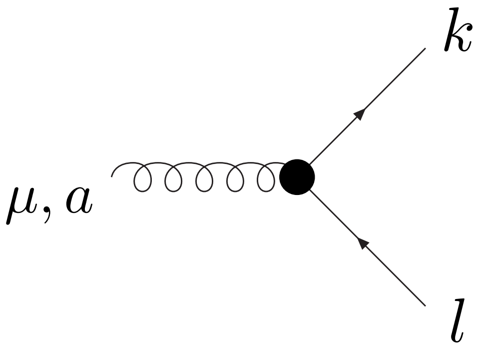
$$\int d^d l \frac{\tilde{l}^2}{\bar{D}_i \bar{D}_j \bar{D}_k \bar{D}_l} = -\frac{i\pi^2}{6} + \mathcal{O}(\varepsilon)$$

R_2 Feynman rules

- In a renormalizable theory, only up to 4-point integrals contribute to the R_2 terms
- They can be included in the computation using special Feynman rules (as it is done for the UV renormalisation). For example:



$$= \frac{ig^2}{16\pi^2}\frac{N_{col}^2-1}{2N_{col}}\delta_{kl}(-\not{p}+2m_q)\,\lambda_{HV}$$



$$= \frac{ig^3}{16\pi^2}\frac{N_{col}^2-1}{2N_{col}}\,t^a_{kl}\gamma_\mu\,(1+\lambda_{HV})$$

Draggiotis, Garzelli, Papadopoulos, Pittau, arXiv:0903.0356

- Similarly to the UV counterterms, the R_2 terms are model dependent and need to be explicitly computed for BSM models This is now automated for renormalizable theories

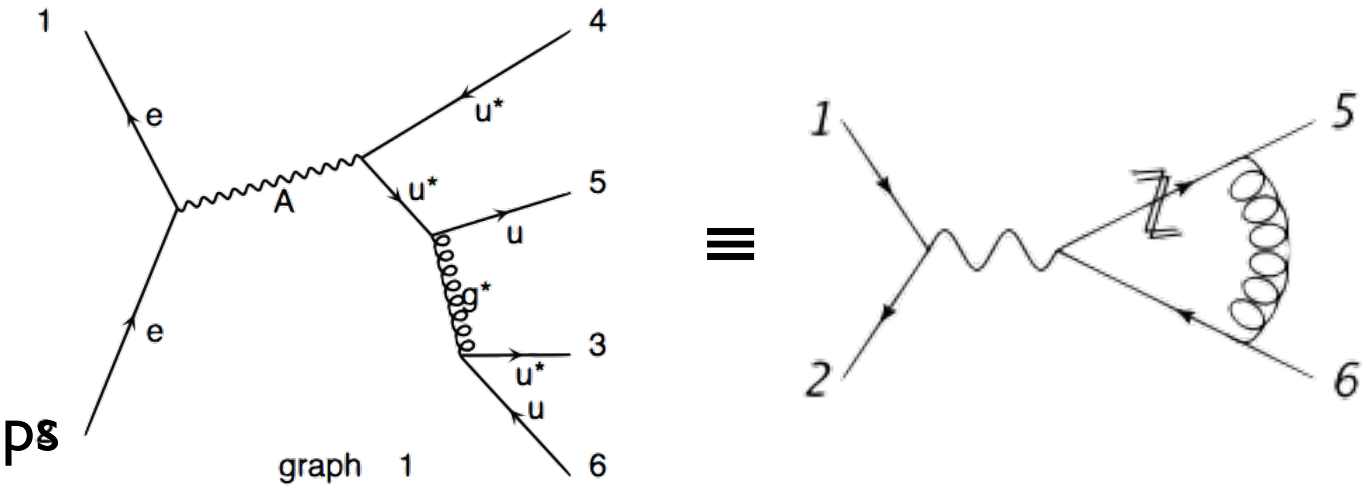
Degrande, arXiv:1406.3030

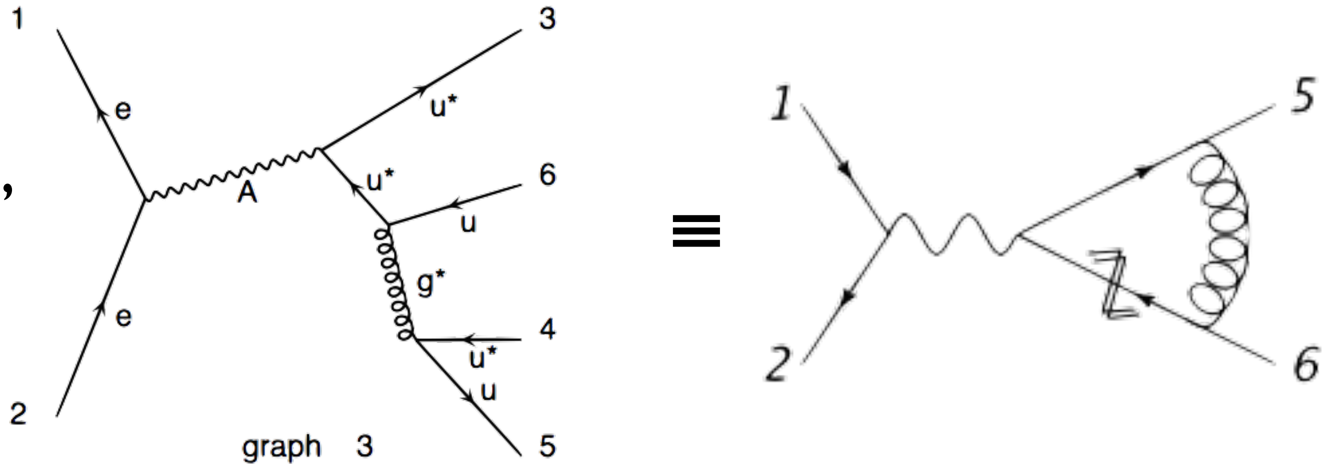
MadLoop

Hirschi et al, arXiv:1103.0621

- How to automate loop computation?
- Exploit MadGraph's capabilities to generate tree-level diagrams
- Loop diagrams with n external legs can be cut, leading to tree diagrams with n+2 legs

- All diagrams with 2 extra particles are generated, those which are needed are filtered out
- Each diagram is assigned a tag, which helps removing mirror/cyclic configurations
- Additional filters to remove tadpole/ bubbles on external legs
- Contract with Born, do the color algebra, re-glue the cut particle, etc…
- Add UV and R2 counterterms as extra vertices





Renormalisation

a focus on μ_r dependence

Going back to renormalisation:

Once diagrams are calculated, one needs to eliminate UV divergencies.
In a renormalisable theory a finite set of renormalisation conditions is sufficient.

We will see in more detail for the EW case, but the idea is the following. For automatising the renormalisation procedure one has to implement a set of new Feynman rules corresponding to UV counterterms, such that UV divergencies are canceled. These terms depend on the renormalisation of physical parameters ($\delta^{\mathrm{UV}}\alpha_s, \delta^{\mathrm{UV}}m_\psi$) and wave-function renormalisation ($\delta^{\mathrm{UV}}Z_g, \delta^{\mathrm{UV}}Z_\psi$).

Now I want to focus to the case of $\delta^{\mathrm{UV}}\alpha_s$, which is typically defined in the so-called $\overline{MS}$ scheme. The philosophy behind it is precisely a Minimal Subtraction , i.e., removing the $1/\epsilon$ pole and promote the *regularisation* scale to the *renormalisation* scale.

$$\delta^{\mathrm{UV}}\alpha_s = -\frac{\alpha_s}{4\pi}\beta_0\Delta \quad \text{with } \Delta = 1/\epsilon - \gamma + \log(4\pi) \ \text{ and } \ \beta_0 = 11 - \frac{2}{3}n_f$$

α_s renormalisation

For a tree-level amplitude $\mathcal{M}_{\rm tree}$ factorising a power α_s^n, the UV part related to the renormalisation of α_s in the corresponding one-loop amplitude, $\mathcal{M}_{1-\rm loop}^{\alpha_s-\rm UV}$, has a form of the kind:

$$\mathcal{M}_{1-\rm loop}^{\alpha_s-\rm UV} = \sim \mathcal{M}_{\rm tree} \times n\frac{\alpha_s}{4\pi}\beta_0\left(\boxed{\Delta + \log(\mu_R^2/Q^2)}\right)$$

together with Δ there is the log of μ_R and another physical scale Q

So then if I look to the renormalised one-loop amplitude $\mathcal{M}_{1-\rm loop}^{\rm ren}$ it contains a term of the form

$$\mathcal{M}_{1-\rm loop}^{\alpha_s-\rm UV} + \mathcal{M}_{\rm tree} n\delta^{UV}\alpha_s = \mathcal{M}_{\rm tree} \times n\frac{\alpha_s}{4\pi}\beta_0\log(\mu_R^2/Q^2)$$

Note that since it is $\mathcal{M}$ and not $|\mathcal{M}|^2$, n can be half-integer

But remember that we have assumed $\mathcal{M}_{\rm tree}$ proportional to α_s^n and

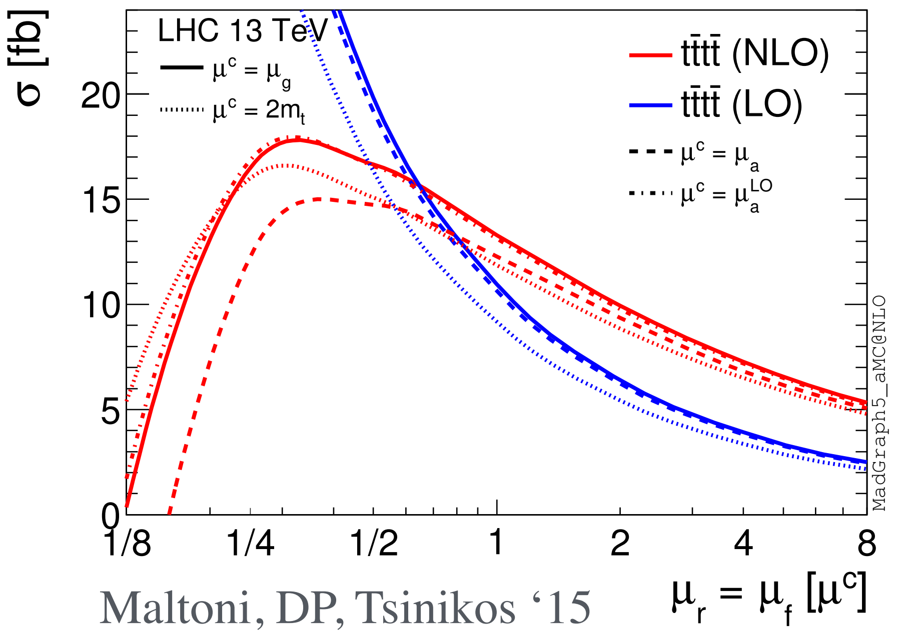
$$\left(\alpha_s(\mu_R)\right)^n = \left(\frac{\alpha_s(\mu_0)}{1+\frac{\alpha_s}{4\pi}\beta_0\log(\mu_R^2/\mu_0^2)}\right)^n \sim \left(\alpha_s(\mu_0)\right)^n \times \left(1 - n\frac{\alpha_s}{4\pi}\beta_0\log(\mu_R^2/\mu_0^2)\right)$$

Remember we are looking to NLO, so terms of $\mathcal{O}(\alpha_s^{n+2})$ are beyond our accuracy target!

α_s renormalisation

$\mathscr{M}^{\text{ren}}_{1-\text{loop}}$ contains terms that cancel the leading dependence of $\mathscr{M}_{\text{tree}}$ on μ_R.

NLO QCD corrections $(\sim 2\Re(\mathscr{M}_{\text{tree}}\mathscr{M}^*_{1-\text{loop}}))$ cancel the leading dependence of LO $(\sim|\mathscr{M}_{\text{tree}}|^2)$ on μ_R.



Maltoni, DP, Tsinikos '15

For instance: $pp \rightarrow t\bar{t}t\bar{t} \sim \alpha_s^4$ at LO.

By varying μ_R by a factor of 2 (this is the HEP-ph dogma) up/down, the scale uncertainty moves from:

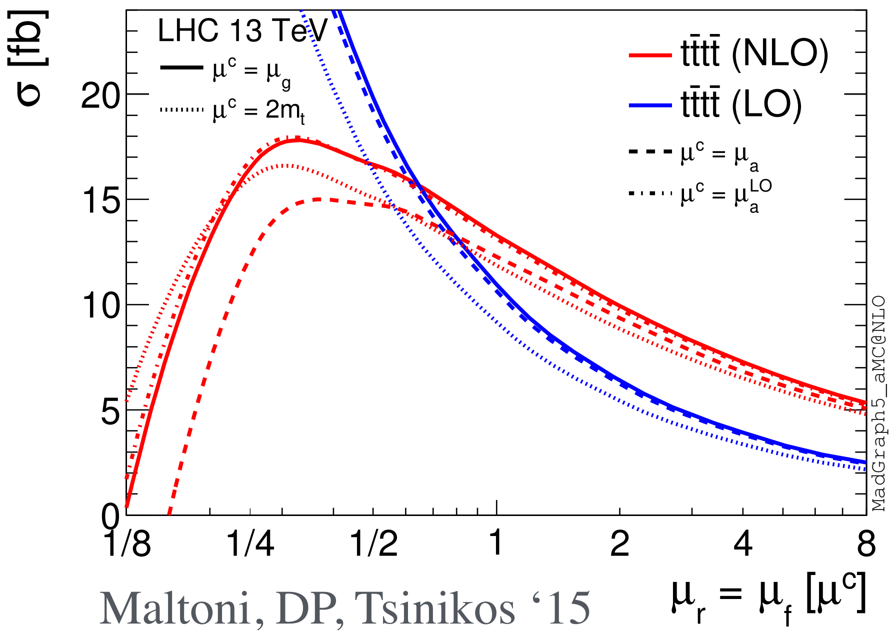
-40% +80% at LO to

-25% +25% at NLO!

α_s renormalisation

$\mathscr{M}^{\text{ren}}_{1-\text{loop}}$ contains terms that cancel the leading dependence of $\mathscr{M}_{\text{tree}}$ on μ_R.

NLO QCD corrections ($\sim 2\Re(\mathscr{M}_{\text{tree}}\mathscr{M}^*_{1-\text{loop}})$) cancel the leading dependence of LO ($\sim|\mathscr{M}_{\text{tree}}|^2$) on μ_R.



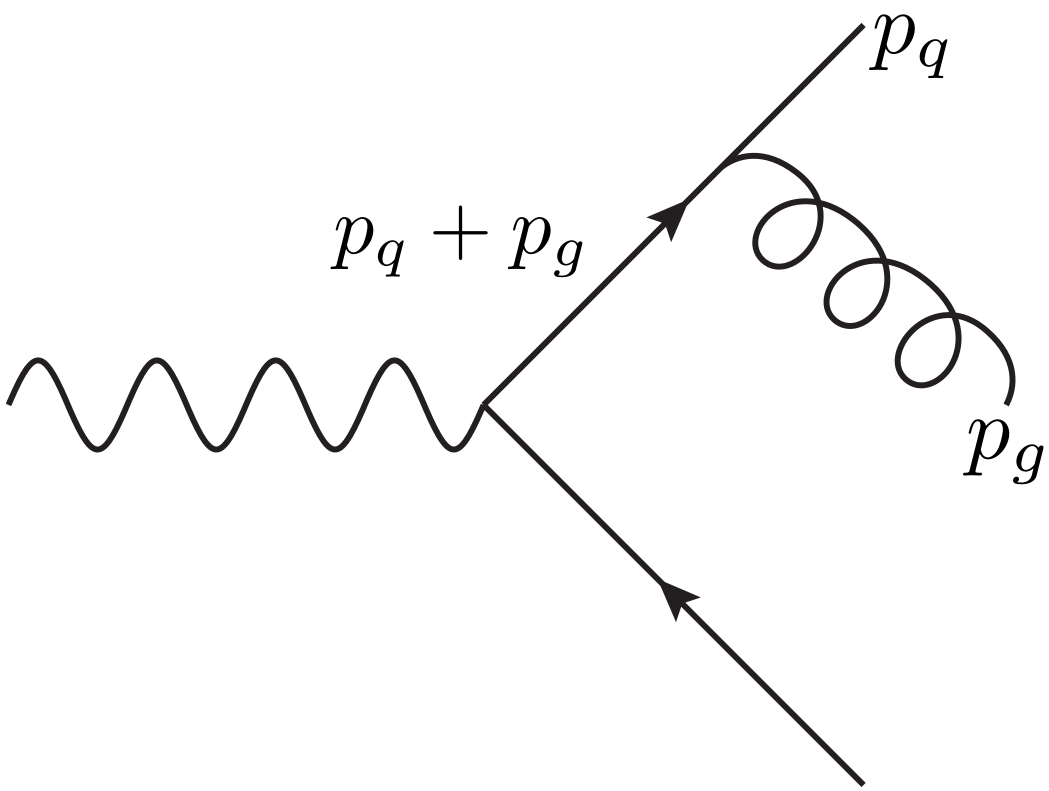
Maltoni, DP, Tsinikos '15

Actually, it is a bit more complex.
This plot depends also on the factorisation scale, but for discussing this we need to look before at IR divergencies.

Infrared divergencies

how to eliminate them

Where do IR divergencies come from?



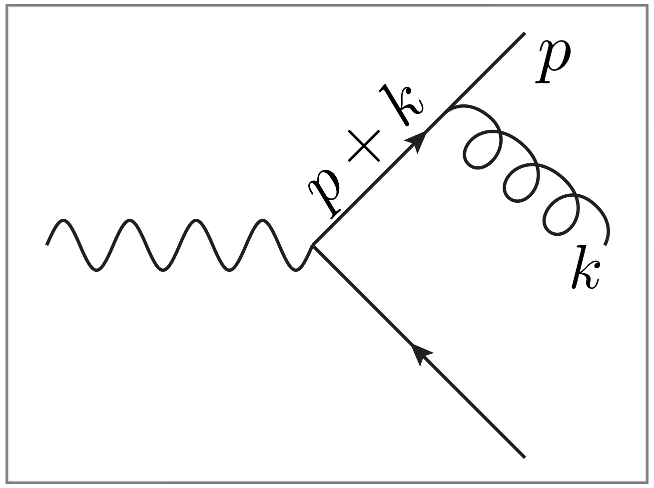
$$\sim \frac{1}{(p_q + p_g)^2 - m_q^2} = \frac{1}{2p_q p_g} =$$

$$= \frac{1}{2E_q E_g \left(1 - \sqrt{1 - m_q^2/E_q^2}\cos(\theta_{gq})\right)}$$

This expression diverges if:
$E_g \to 0$: soft divergence
and or $\theta_{gq} \to 0$ if $m_q = 0$: collinear divergence

If the p_g momentum is integrated in $d = 4 - 2\epsilon$ dimensions, these configurations lead to $1/\epsilon$ poles and even $1/\epsilon^2$ poles in the case $m_q = 0$ and $E_g \to 0$ and $\theta_{gq} \to 0$.

Divergencies have a “simple” structure



$$(p+k)^2 = 2E_pE_k(1-\cos\theta_{pk})$$

- Collinear singularity:

$$\lim_{p//k} |M_{n+1}|^2 \simeq |M_n|^2 \; P^{AP}(z)$$

- Soft singularity:

$$\lim_{k\to 0} |M_{n+1}|^2 \simeq \sum_{ij} |M_n^{ij}|^2 \frac{p_i p_j}{p_i k \; p_j k}$$

27

For instance, being $z = (p+k)/p$ in the collinear and massless limit:

$$P_{qq}(z) = C_F \frac{1+z^2}{1-z}$$

Therefore knowing the Born amplitude, the coefficient in front of the $1/\epsilon$ poles is also known, since only the integrals of P_{XY} and $\frac{p_i p_j}{p_i k \; p_j k}$ have to be calculated.

Cancellations between reals and loops

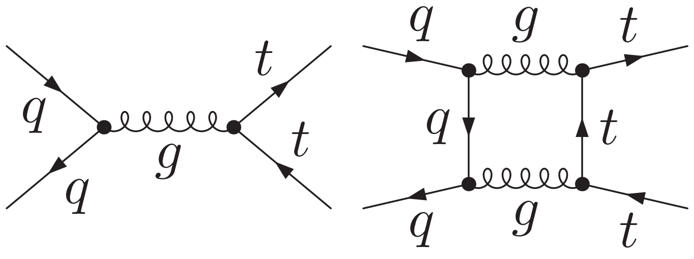
According to the Kinoshita-Lee-Nauenberg (KLN) theorem:
If you are inclusive over the possible final states, the IR divergencies related the final state (FS) cancel:

LOOP+REALS at a given $\mathcal{O}(\alpha_s^n)$ => no IR FS divergencies at $\mathcal{O}(\alpha_s^n)$

Example:

$q\bar{q} \to t\bar{t}$

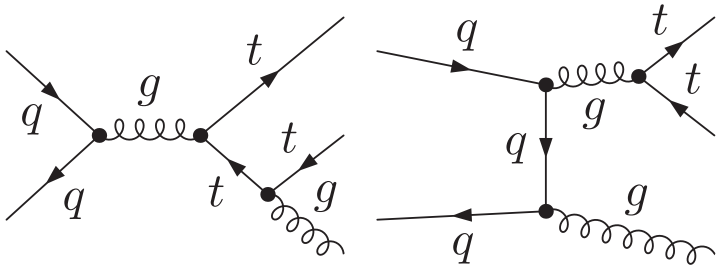
$\overline{|\mathcal{M}^{t\bar{t}}|}^2_{\mathcal{O}(\alpha_s^3)}$



The interference of these 2 diagrams has soft divergencies,

$q\bar{q} \to t\bar{t}g$

$\overline{|\mathcal{M}^{t\bar{t}g}|}^2_{\mathcal{O}(\alpha_s^3)}$



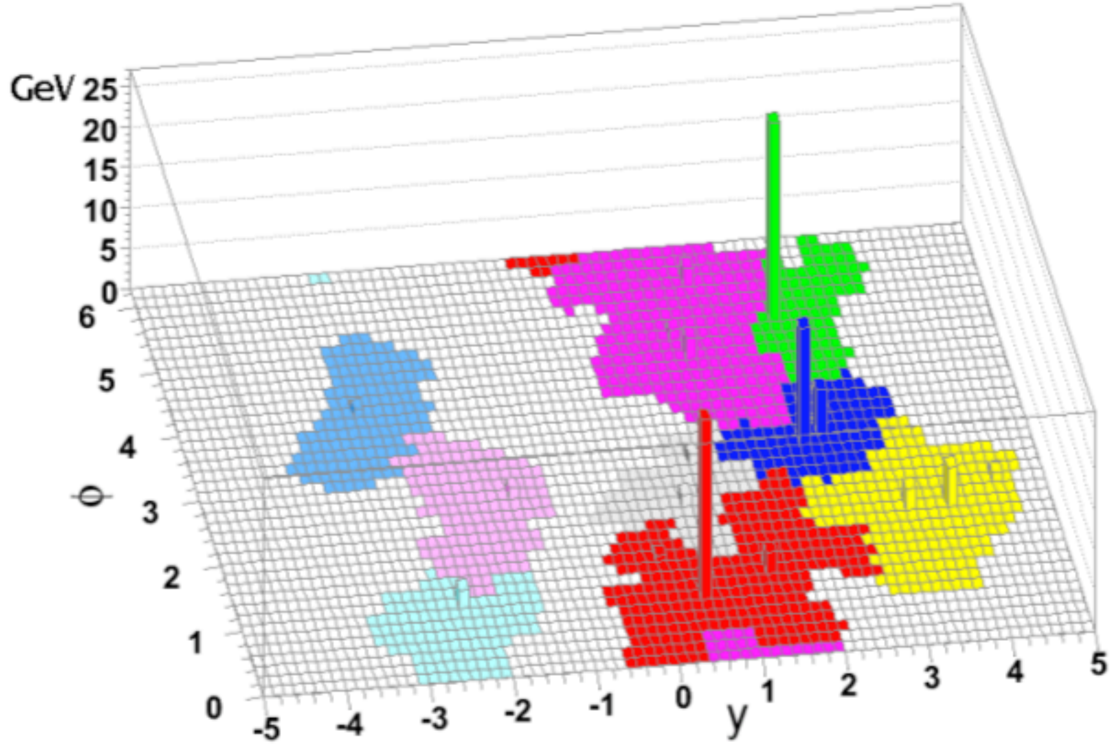
which cancel exactly against the interference of these two diagrams…

…if one looks at ***inclusive*** $t\bar{t}$ production

IR safety

I cannot require the calculation of the cross section of “exactly the $t\bar{t}$ ” final state: it is not IR safe.
Indeed, the presence of an infinitesimally soft gluon would change the categorisation of the event —> DIVERGENT quantity.

The same story for the cross section of $t\bar{t}q$ production, where q is a light quark. The presence of an infinitesimally soft, or hard and collinear, gluon would lead to the same problem.



Jets observables are precisely IR-safe quantities that deals with the hard radiation of light colored particles.

Jets are IR-safe final-state objects, while partons are not!

… still the divergencies do not cancel for the initial state

It looks bad, but it is actually the reason why the dependence on the factorisation scale diminishes. Let’s see why:

$$t = \log \frac{Q^2}{\mu^2}$$

$$\frac{df_q(x,t)}{dt} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{qq}(z)\, f_i(\frac{x}{z},t) \,+\, P_{qg}(z) f_g(\frac{x}{z},t)\right]$$

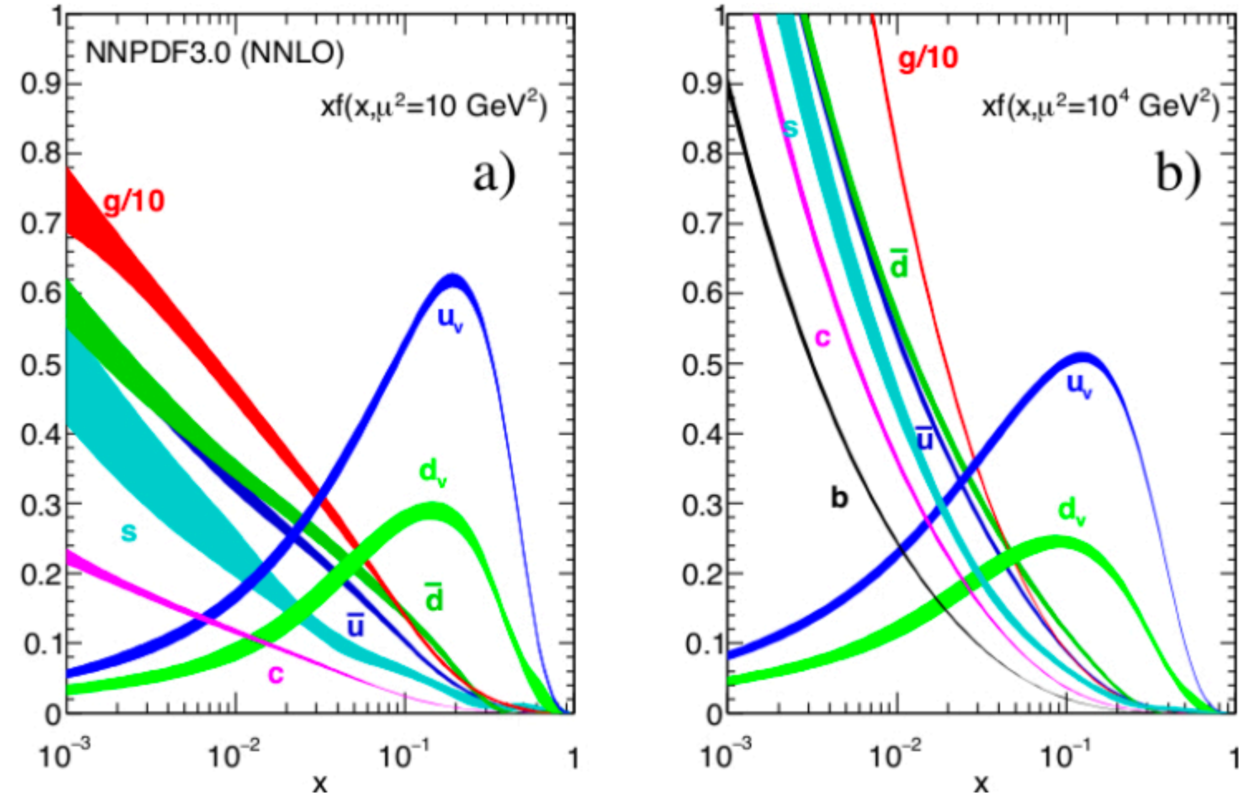
$$\frac{df_g(x,t)}{dt} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{gq}(z)\sum_{i=q,\bar{q}} f_i\left(\frac{x}{z},t\right) + P_{gg}(z) f_g\left(\frac{x}{z},t\right)\right]$$

DGLAP equation for PDFs:

$$P_{qg} = \frac{1}{2}\left[z^2 + (1-z)^2\right]$$

$$P_{gq}(z) = P_{qq}(1-z) = C_F\frac{1+(1-z)^2}{z}$$

$$P_{gg}(z) = 2C_A\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right]$$



Parton Distribution Functions (PDFs), and consequently the predictions for hadroproduction cross sections, depend on the factorisation scale μ_F via the DGLAP equations.

reduction of μ_F dependence

The partial cancellation that we have observed for μ_R via the renormalisation takes place also in the case of μ_F via PDFs.

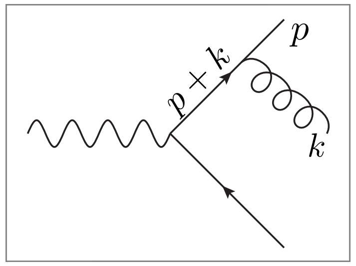
PDF evolution:

$$\frac{df_q(x,t)}{dt} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{qq}(z)\, f_i(\frac{x}{z},t) + P_{qg}(z) f_g(\frac{x}{z},t)\right]$$

$$\frac{df_g(x,t)}{dt} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{gq}(z)\sum_{i=q,\bar{q}} f_i\left(\frac{x}{z},t\right) + P_{gg}(z) f_g\left(\frac{x}{z},t\right)\right]$$

$$t = \log\frac{Q^2}{\mu^2}$$

Singularities structure:



$$(p+k)^2 = 2E_pE_k(1-\cos\theta_{pk})$$

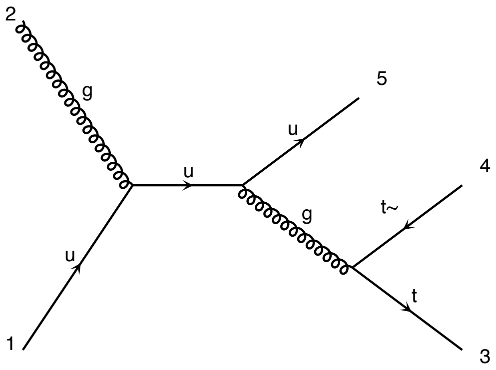
- Collinear singularity:

$$\lim_{p//k}|M_{n+1}|^2 \simeq |M_n|^2\; P^{AP}(z)$$

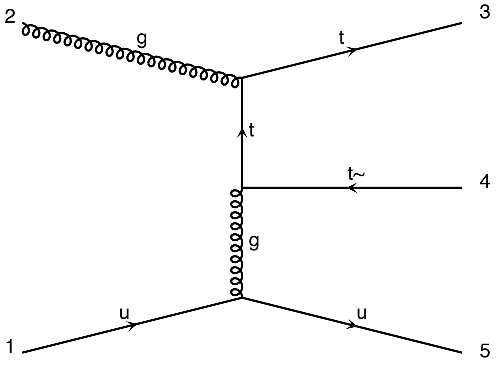
- Soft singularity:

$$\lim_{k\to 0}|M_{n+1}|^2 \simeq \sum_{ij}|M_n^{ij}|^2\frac{p_ip_j}{p_ik\; p_jk}$$

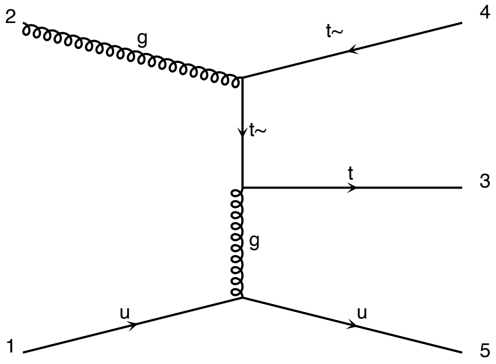
27



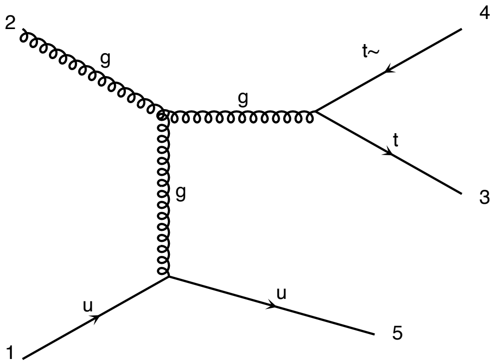
real diagram 1 QCD=3, QED=0



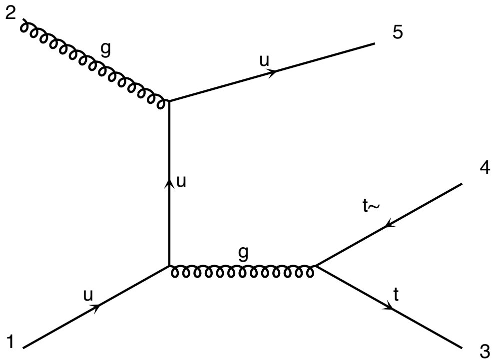
real diagram 2 QCD=3, QED=0



real diagram 3 QCD=3, QED=0



real diagram 4 QCD=3, QED=0



real diagram 5 QCD=3, QED=0

$$\mathscr{M}_{qg\to\bar{t}tq}$$

reduction of μ_F dependence

The partial cancellation that we have observed for μ_R via the renormalisation takes place also in the case of μ_F via PDFs.

PDF evolution:

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$$t = \log\frac{Q^2}{\mu^2}$$

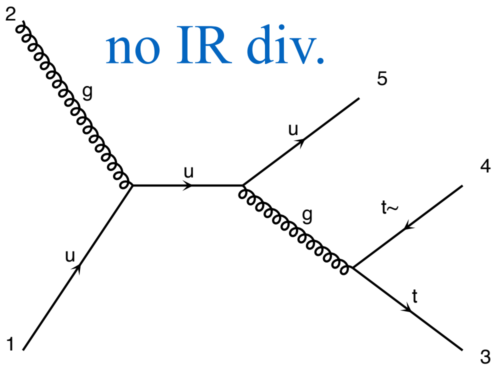
Singularities structure:

Depending on the kinematical condition:

$|\mathscr{M}_{q_i g\to t\bar{t}q_f}|^2 \sim |\mathscr{M}_{q\bar{q}\to t\bar{t}}|^2 P_{qg}$ for $q_f \parallel g$

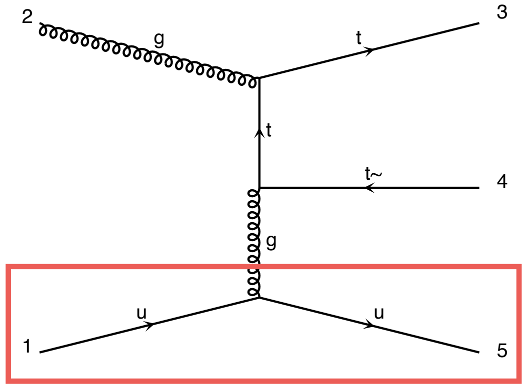
$|\mathscr{M}_{q_i g\to t\bar{t}q_f}|^2 \sim |\mathscr{M}_{gg\to t\bar{t}}|^2 P_{gq}$ for $q_f \parallel q_i$

Together with $1/\epsilon$ poles, collinear divergencies exhibit $\log(\mu^2/Q^2)$ terms factorising P_{XY}.
They need to be subtracted at the scale $\mu = \mu_F$, and their μ_F dependence cancel exactly the $\mathcal{O}(\alpha_s)$ dependence on μ_F from PDF evolution.

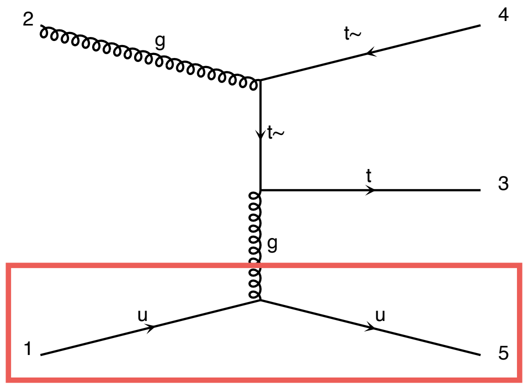


no IR div.

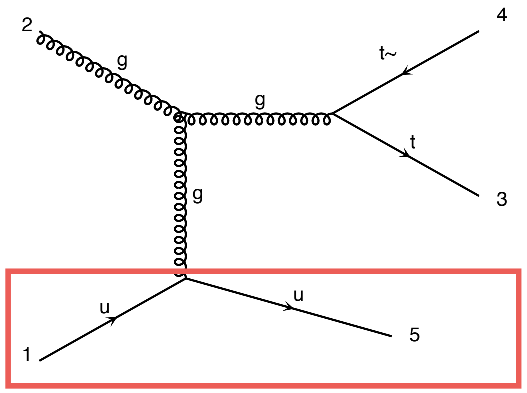
real diagram 1 QCD=3, QED=0



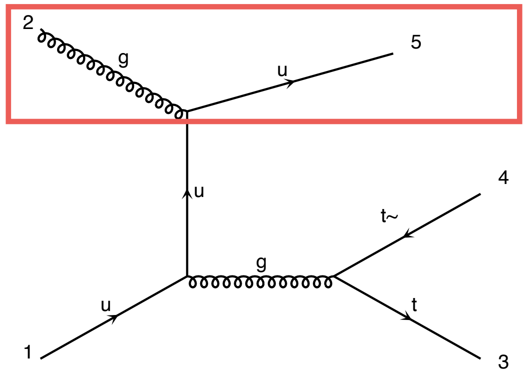
real diagram 2 QCD=3, QED=0



real diagram 3 QCD=3, QED=0



real diagram 4 QCD=3, QED=0



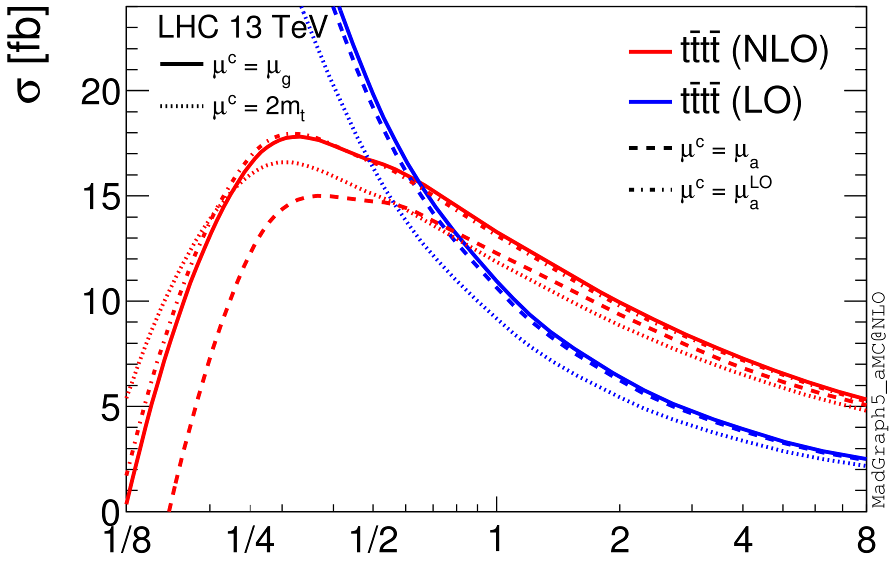
real diagram 5 QCD=3, QED=0

$\mathscr{M}_{qg\to\bar{t}tq}$

μ_F and μ_R dependence form LO to NLO (revisited)

$\mathscr{M}^{\text{ren}}_{1-\text{loop}}$ contains terms that cancel the leading dependence of $\mathscr{M}_{\text{tree}}$ on μ_R and $|\mathscr{M}|^2_{\mathcal{O}(\alpha_s^{n+1})}$ for real radiation (and loop) contain terms that cancel the PDF evolution of the PDFs associated to $|\mathscr{M}_{\text{tree}}|^2_{\mathcal{O}(\alpha_s^{n})}$ at Born.

NLO QCD corrections cancel the leading dependence of LO on μ_R and μ_F.



Maltoni, DP, Tsinikos '15

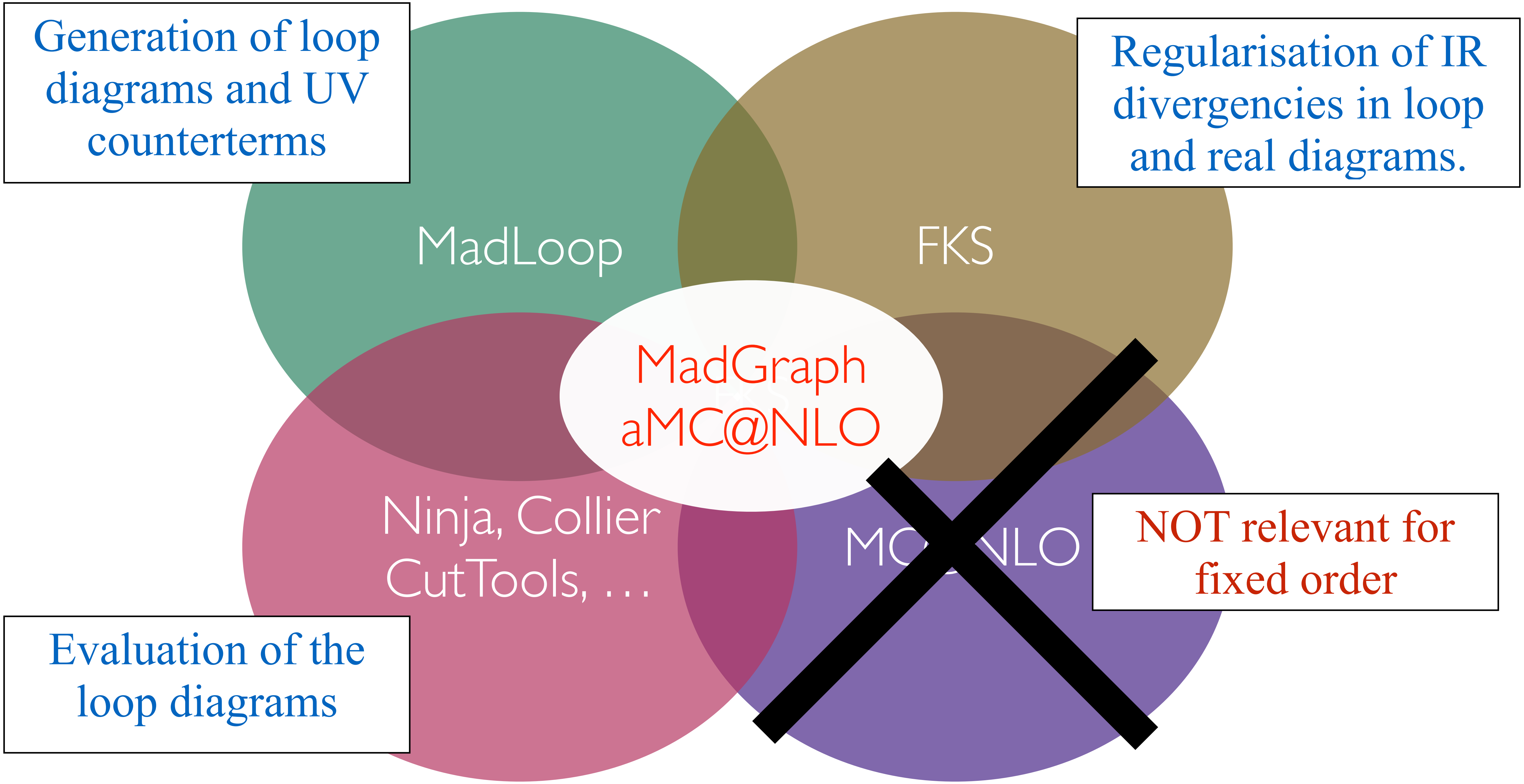
For instance: $pp \to t\bar{t}t\bar{t} \sim \alpha_s^4$ at LO.

By varying $\mu_R = \mu_F$ by a factor of 2 up/down, the scale uncertainty moves from:
-40%/+80% at LO to
-25%/+25% at NLO!

Automation at NLO QCD:

how to put things together

The structure of MadGraph resembles the steps of a calculation that in principle one could do with pencil and paper:



Generation of loop diagrams and UV counterterms

Regularisation of IR divergencies in loop and real diagrams.

MadLoop

FKS

MadGraph
aMC@NLO

Ninja, Collier
CutTools, …

~~MC@NLO~~

NOT relevant for fixed order

Evaluation of the loop diagrams

Alwall, Frederix, Frixione, Hirschi, Maltoni, Mattelaer, Shao, Stelzer, Torrielli, Zaro ’14
Frederix, Frixione, Hirschi, DP, Shao, Zaro ‘18

Phase space integration

$$\sigma_{NLO} = \int d^4\Phi_n \mathcal{B} + \int d^4\Phi_n \mathcal{V} + \int d^4\Phi_{n+1} \mathcal{R}$$

contains $\int d^d l$

- For complicated processes the integrations have to be done via MonteCarlo techniques, in an integer number of dimensions
- Divergences have to be canceled explicitly
- Slicing/Subtraction methods have been developed to extract divergences from the phase-space integrals

Again slides stolen from Marco Zaro, who gave this lecture at the 2022 edition of this school. More details can be found on this particular subject in his slides

Example

- Suppose that we can cast the phase space integral in the form

$$\int_0^1 dx f(x) \quad \text{with} \quad f(x) = \frac{g(x)}{x} \quad \text{and } g(x) \text{ a regular function}$$

- We introduce a regulator which renders the integral finite

$$\int_0^1 dx x^{\varepsilon} f(x) = \int_0^1 dx \frac{g(x)}{x^{1-\varepsilon}}$$

- The divergence will turn into a pole in ε. How can we extract the pole?

Phase space slicing

$$\lim_{\varepsilon\to 0}\int_0^1 dx x^\varepsilon f(x) = \lim_{\varepsilon\to 0}\int_0^1 dx \frac{g(x)}{x^{1-\varepsilon}}$$

- We introduce a small parameter δ≪1:

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^1 dx \frac{g(x)}{x^{1-\varepsilon}} &= \lim_{\varepsilon\to 0}\left(\int_0^\delta dx \frac{g(x)}{x^{1-\varepsilon}} + \int_\delta^1 dx \frac{g(x)}{x^{1-\varepsilon}}\right)\\ &\simeq \lim_{\varepsilon\to 0}\left(\int_0^\delta dx \frac{g(0)}{x^{1-\varepsilon}} + \int_\delta^1 dx \frac{g(x)}{x^{1-\varepsilon}}\right)\\ &= \lim_{\varepsilon\to 0}\frac{\delta^\varepsilon}{\varepsilon}g(0) + \int_\delta^1 dx \frac{g(x)}{x}\\ &= \lim_{\varepsilon\to 0}\left(\boxed{\frac{1}{\varepsilon}} + \log\delta\right)g(0) + \boxed{\int_\delta^1 dx \frac{g(x)}{x}}\end{aligned}$$

finite integral
(can be computed numerically)

pole in ε

Subtraction method

$$\lim_{\varepsilon \to 0} \int_0^1 dx x^\varepsilon f(x) = \lim_{\varepsilon \to 0} \int_0^1 dx \frac{g(x)}{x^{1-\varepsilon}}$$

- Add and subtract $g(0)/x$

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_0^1 dx \frac{g(x)}{x^{1-\varepsilon}} &= \lim_{\varepsilon \to 0} \int_0^1 dx x^\varepsilon \left(\frac{g(0)}{x} + \frac{g(x)}{x} - \frac{g(0)}{x} \right) \\ &= \lim_{\varepsilon \to 0} \int_0^1 dx \left(\frac{g(0)}{x^{1-\varepsilon}} + \frac{g(x) - g(0)}{x^{1-\varepsilon}} \right) \\ &= \lim_{\varepsilon \to 0} \boxed{\frac{1}{\varepsilon} g(0)} + \int_0^1 dx \boxed{\frac{g(x) - g(0)}{x}}\end{aligned}$$

pole in ε

finite integral
(can be computed numerically)

Slicing vs Subtraction

- In both cases the pole is extracted and we end up with a finite remainder:

$$g(0)\log\delta + \int_\delta^1 dx\,\frac{g(x)}{x} \qquad \int_0^1 dx\,\frac{g(x)-g(0)}{x}$$

- Subtraction acts like a plus distribution
- Slicing works only for small δ: δ-independence of cross section and distributions must be proven; subtraction is exact
- Both methods have cancelations between large numbers. If for a given observable $\lim_{x\to 0} O(x) \neq O(0)$ or we choose a too small bin size, instabilities will arise (we cannot ask for an infinite resolution)
- Subtraction is in general more flexible: good for automation

NLO with subtraction

$$\sigma_{NLO} = \int d^4\Phi_n \mathcal{B} + \int d^4\Phi_n \mathcal{V} + \int d^4\Phi_{n+1} \mathcal{R}$$

- With the subtraction terms the expression becomes

$$\begin{aligned}\sigma_{NLO} = &\int d^4\Phi_n \mathcal{B} \\ &+ \int d^4\Phi_n \left(\mathcal{V} + \int d^d\Phi_1 \mathcal{C}\right)_{\varepsilon\to 0} \\ &+ \int d^4\Phi_{n+1} \left(\mathcal{R} - \mathcal{C}\right)\end{aligned}$$

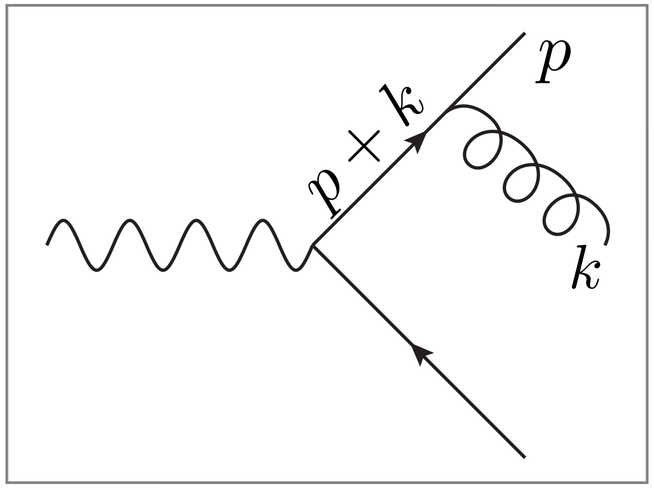
Poles cancel from *d*-dim integration

Integrand is finite in 4 dimension

- Terms in brackets are finite and can be integrated numerically in *d*=4 and independently one from another

The subtraction term

- The subtraction term C should be chosen such that:
  - It exactly matches the singular behaviour of R
  - It can be integrated numerically in a convenient way
  - It can be integrated exactly in d dimension, leading to the soft and/or collinear poles in the dimensional regulator
  - It is process independent (overall factor times Born)
- QCD comes to help: structure of divergences is universal:



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- Soft singularity:

$$\lim_{k\to 0} |M_{n+1}|^2 \simeq \sum_{ij} |M_n^{ij}|^2 \frac{p_i p_j}{p_i k \; p_j k}$$

Automation at NLO QCD:

some results you can obtain with MadGraph

Some results for $t\bar{t}V$ processes

13 TeV σ[fb]	$t\bar{t}H$	$t\bar{t}Z$	$t\bar{t}W^{\pm}$	$t\bar{t}\gamma$
NLO	$522.2^{+6.0\%\ +2.1\%}_{-9.4\%\ -2.6\%}$	$873.6^{+10.3\%\ +2.0\%}_{-11.7\%\ -2.5\%}$	$644.8^{+13.0\%\ +1.7\%}_{-11.6\%\ -1.3\%}$	$2746^{+14.2\%\ +1.6\%}_{-13.5\%\ -1.9\%}$
LO	$476.6^{+35.5\%\ +2.0\%}_{-24.2\%\ -2.1\%}$	$710.3^{+36.1\%\ +2.0\%}_{-24.5\%\ -2.1\%}$	$526.9^{+28.1\%\ +1.7\%}_{-20.4\%\ -1.8\%}$	$2100^{+36.2\%\ +1.8\%}_{-24.5\%\ -1.9\%}$
K-factor	1.10	1.23	1.22	1.31